A SHORT ELEMENTARY PROOF OF REVERSED BRUNN–MINKOWSKI INEQUALITY FOR COCONVEX BODIES

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Abstract. The theory of coconvex bodies was formalized by A. Khovanskiĭ and V. Timorin in [KT14]. It has fascinating relations with the classical theory of convex bodies, as well as applications to Lorentzian geometry. In a recent preprint [Sch17], R. Schneider proved a result that implies a reversed Brunn–Minkowski inequality for coconvex bodies, with description of equality case. In this note we show that this latter result is an immediate consequence of a more general result, namely that the volume of coconvex bodies is strictly convex. This result itself follows from a classical elementary result about the concavity of the volume of convex bodies inscribed in the same cylinder.

Let C be a closed convex cone in $\mathbb{R}^n$, with non empty interior, and not containing an entire line. A *C-coconvex body* K is a non-empty closed bounded proper subset of C such that $C \setminus K$ is convex. The set of C-coconvex bodies is stable under positive homotheties. It is also stable for the $\oplus$ operation, defined as $K_1 \oplus K_2 = C \setminus (C \setminus K_1 + C \setminus K_2)$, where $+$ is the Minkowski sum. The following reversed Brunn–Minkowski theorem is proved in [Sch17] (see [KT14] for a partial result). We denote by V_n the volume in $\mathbb{R}^n$.

Theorem 1. *Let K_1, K_2 be C-coconvex bodies, and $\lambda \in (0,1)$. Then*

$$V_n((1-\lambda)K_1 \oplus \lambda K_2)^{1/n} \leq (1-\lambda)V_n(K_1)^{1/n} + \lambda V_n(K_2)^{1/n} ,$$

and equality holds if and only if $K_1 = \alpha K_2$ for some $\alpha > 0$.

Remark 2. What is actually proved in [Sch17] in the analogous of Theorem 1 for *C-coconvex sets* instead of C-coconvex bodies: the set is not required to be bounded but only to have finite Lebesgue measure. So the result of [Sch17] requires a more involved proof than the one presented here.

Actually, we will see that the following result holds.

Theorem 3. *The volume is strictly convex on the set of C-coconvex bodies. More precisely, if K_1, K_2 are C-coconvex bodies, and $\lambda \in (0,1)$, then*

$$V_n((1-\lambda)K_1 \oplus \lambda K_2) \leq (1-\lambda)V_n(K_1) + \lambda V_n(K_2) .$$

Moreover, equality holds if and only if $K_1 = K_2$.

The following elementary lemma, together with the fact that V_n is positively homogeneous of degree n (i.e. $V_n(tA) = t^n V_n(A)$ for $t > 0$), shows that Theorem 3 implies Theorem 1.

Lemma 4. *Let f be a positive convex function, positively homogeneous of degree n. Then $f^{1/n}$ is convex.*

Suppose moreover that f is strictly convex. If there is $\lambda \in (0,1)$ such that $f^{1/n}((1-\lambda)x+\lambda y))$ equals $(1-\lambda)f^{1/n}(x) + \lambda f^{1/n}(y)$, then there is $\alpha > 0$ with $x = \alpha y$.

Proof. For $\bar{\lambda} \in [0,1]$ and any x, y, we have $f((1-\bar{\lambda})\frac{x}{f(x)^{1/n}} + \bar{\lambda}\frac{y}{f(y)^{1/n}}) \leq 1$, and the result follows by taking, for any $\lambda \in (0,1)$, $\bar{\lambda} = \lambda f(y)^{1/n}/((1-\lambda)f(x)^{1/n} + \lambda f(y)^{1/n})$. □

Let us prove Theorem 3.

Let H be an affine hyperplane of $\mathbb{R}^n$ with the following properties: it has an orthogonal direction in the interior of C, K_1, K_2 and the origin are contained in the same half-space H^+ bounded by H, and $H \cap C = B$ is compact. For $\lambda \in [0,1]$, let $K_\lambda = (1-\lambda)K_1 \oplus \lambda K_2$, which is also contained in H^+, and let $\mathrm{cap}_H(K_\lambda) = H^+ \cap (C \setminus K_\lambda)$, see Figure 1.

Date: November 12, 2017.
The author thanks Ivan Izmestiev and Rolf Schneider.
Keywords: coconvex sets, covolume, Brunn–Minkowksi.

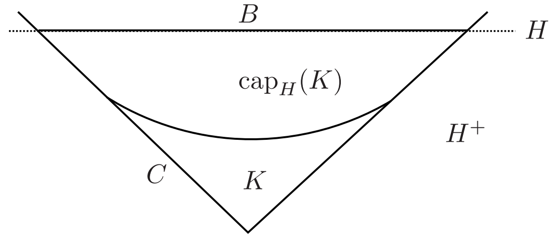

FIGURE 1. Notations

Also, the quantity $V_n(K_\lambda) + V_n(\mathrm{cap}_H(K_\lambda))$ does not depend on λ, as it is equal to $V_n(C \cap H^+)$. Hence Theorem 3 is equivalent to

$$V_n(\mathrm{cap}_H(K_\lambda)) \geq (1-\lambda)V_n(\mathrm{cap}_H(K_1)) + \lambda V_n(\mathrm{cap}_H(K_2))$$

for $\lambda \in (0,1)$, with equality if and only if $K_1 = K_2$.

This last result itself follows from the following elementary result. Here "elementary" means that the most involved technique in its proof is Fubini theorem (see Chapter 50 in [BF87] or Lemma 3.30 in [BF17]).

Lemma 5. *Let A_0 and A_1 be two convex bodies in $\mathbb{R}^n$ contained in H^+, such that their orthogonal projection onto H is B. Then, for $\lambda \in [0,1]$,*

$$V_n((1-\lambda)A_0 + \lambda A_1) \geq (1-\lambda)V_n(A_0) + \lambda V_n(A_1) \ .$$

Equality holds if and only if either $A_0 = A_1 + U$ or $A_1 = A_0 + U$, where U is some segment whose direction is orthogonal to H.

In our case, if K is a C-coconvex body, then $K \oplus U$ is a C-coconvex body if and only if $U = \{0\}$.

Remark 6. In the classical convex bodies case, the Brunn–Minkowski inequality (saying that the nth-root of the volume of convex bodies is concave) follows from the more general result that the volume of convex bodies is log-concave. This is the genuine analogue of our situation, due to the following implications:

$$f \text{ concave} \implies f \text{ log-concave}$$
$$f \text{ log convex} \implies f \text{ convex} \ .$$

If moreover f is positively homogenous of degree n, we have:

$$f \text{ log-concave} \implies f^{1/n} \text{ concave}$$
$$f \text{ convex} \implies f^{1/n} \text{ convex} \ .$$

Remark 7. Actually we didn't use the fact that the convex set C is a cone, as the only thing that really matters is the stability of C-coconvex bodies under convex combinations. See e.g. [BF17] for an application to this more general situation. If C is a cone, the C-coconvex bodies are furthermore stable under positive homotheties and $\oplus$, that allows to develop a mixed-volume theory for C-coconvex sets, see [Fil13, KT14, Sch17].

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