

# Quasi-Fuchsian co-Minkowski manifolds

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## Abstract

This survey is an introduction to the geometry of co-Minkowski space, the space of unoriented spacelike hyperplanes of the Minkowski space. Affine deformations of cocompact lattices of hyperbolic isometries act on it, in a way similar to the way that quasi-Fuchsian groups act on hyperbolic space. In particular, there is a convex core. There is also a unique “mean” hypersurface, i.e. with traceless second fundamental form. The mean distance between the mean hypersurface and the lower boundary of the convex core endows the space of affine deformations of a given lattice with an asymmetric norm. The symmetrization of the asymmetric norm is simply the volume of the convex core.

In dimension  $2+1$ , the asymmetric norm is the total length of the bending lamination of the lower boundary component of the convex core. We obtain an extrinsic proof of a theorem of Thurston saying that, on the tangent space of Teichmüller space, the total length of measured geodesic laminations is an asymmetric norm.

We also exhibit and comment the Anosov-like character of these deformations, similar to the Anosov character of the quasi-Fuchsian representations pointed out in [GW12].

**Keywords**— Co-Minkowski space, compact hyperbolic manifolds, Earthquake norm, Codazzi tensors, convex core, Anosov representation

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# 1 Introduction

**Action of hyperbolic isometries on model spaces** Let  $\mathbb{H}^d/\Gamma$  be an oriented compact hyperbolic manifold. In the Klein projective model, the hyperbolic space  $\mathbb{H}^{d+1}$  is the interior of a ball, and some features of the action of  $\Gamma$  can be described looking at the exterior of the ball, naturally endowed with a Lorentzian structure of constant curvature one, and called de Sitter space. Using affine duality with respect to the unit sphere, de Sitter space can be seen as the space of totally geodesic hypersurfaces of  $\mathbb{H}^d$ .

Since the work of G. Mess [Mes07, ABB<sup>+</sup>07], the action of cocompact lattices of  $O(d, 1)$  on Lorentzian constant curvature model spaces attracted attention from geometers, see e.g. the surveys [FS16, Bar16]. Apart from de Sitter space, Anti-de Sitter space has constant curvature  $-1$  and Minkowski space is the flat one. As we said, de Sitter space is the dual of the hyperbolic space, and Anti-de Sitter space is its own dual, see e.g. [FS18]. Co-Minkowski space is the dual of Minkowski space. More precisely, it is the space of spacelike hyperplanes of Minkowski space. It comes with a degenerate metric of constant curvature one.

In other terms, if one wants to look at the action of subgroups of  $O(d, 1)$  on  $d + 1$  dimensional model spaces<sup>1</sup>, up to duality, it is the same to consider Lorentzian model spaces or constant curvature  $-1$  model spaces:

Curvature $-1$ spaces	$\xleftrightarrow{\text{dual}}$	Lorentzian spaces
Hyperbolic space	$\longleftrightarrow$	de Sitter space
co-Minkowski space	$\longleftrightarrow$	Minkowski space
Anti-de Sitter space	$\longleftrightarrow$	Anti-de Sitter space

**Co-Minkowski space** The first part of this survey is an elementary introduction to co-Minkowski space  ${}^*\mathbb{R}^{d,1}$ . This space has recently attracted attention under the name "half-pipe", as introduced by J. Dancinger in [Dan11, Dan13]<sup>2</sup>, and used in recent works [Sep15, Sch16, DMS14], see also [FS18].

We will focus on a "Klein model" of co-Minkowski space as the subspace  $B^d \times \mathbb{R}$  of the affine space  $\mathbb{R}^{d+1}$ , where  $B^d$  is an open unit ball, see Figure 1. In general, the interest

<sup>1</sup>We call a  $d$ -dimensional *model space* the quotient by the antipodal map of a pseudo-sphere in  $\mathbb{R}^{d+1}$ , see [FS18].

<sup>2</sup>The surface  $coMin^{1+1}$  in Figure 2 would deserve the name *half-pipe*. The name co-Minkowski space comes from the particular situation of this *co-pseudo-Euclidean space*, see the corresponding entry in the *Encyclopædia of Mathematics*.

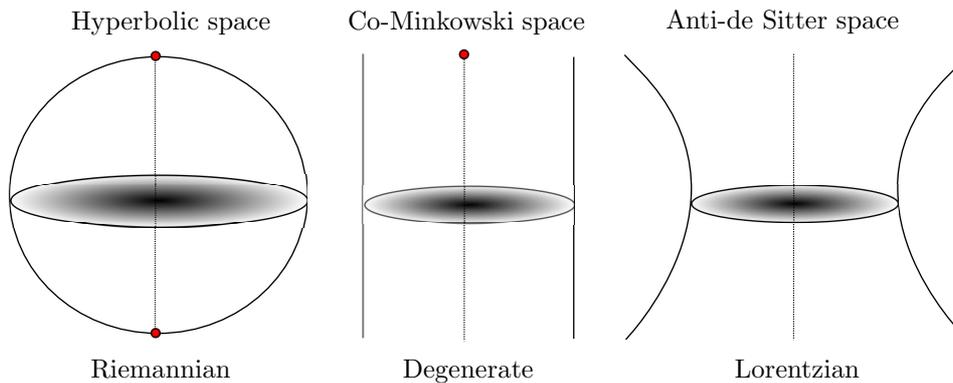


Figure 1: Affine models of the three 3d model spaces of constant curvature  $-1$ . Shaded discs are totally geodesics embedded hyperbolic planes.

of an affine model is that (unparameterized) geodesics are affine segments, so for example some affine notions as convexity or convex hull are easily tractable. In the particular case of co-Minkowski space, many analogues of classical differential geometry results are easier than the original ones, for example:

- the (smooth) hypersurfaces carrying a non-degenerate induced metric are all hyperbolic, and when they are metrically complete, they are graphs of functions on the ball  $B^d$ ,
- the shape operator of graph hypersurfaces gives symmetric Codazzi tensors on the hyperbolic space  $\mathbb{H}^d$ ,
- actually, the correspondence between complete hyperbolic hypersurfaces and hyperbolic symmetric Codazzi tensor is one-to-one, that gives a simplified co-Minkowski version of the fundamental theorem of hypersurfaces (Section 2.3.1),
- complete hyperbolic hypersurfaces such that the trace of the shape operator vanishes are called *mean surfaces*; existence and uniqueness of such hypersurfaces are straightforward consequence of classical theory of elliptic PDE on the ball (Section 2.3.2),
- the functions whose graph is a boundary of the convex hull of the graph of a continuous map  $b : \partial B^d \rightarrow \mathbb{R}$  are solutions of the classical Monge–Ampère equation (Section 2.3.3).

Another nice feature of the cylinder model of co-Minkowski space is that it allows an easy definition of *degenerations* of hyperbolic or Anti-de Sitter manifolds to a co-Minkowski manifold, as Figure 1 heuristically suggests. In turn, co-Minkowski geometry as a *transitional geometry* between the hyperbolic geometry and the AdS geometry was the main motivation of [Dan13, Dan11], see also [Sep15, FS18]. Unfortunately, such considerations are out of the scope of the present survey.

**The action of  $H^1(\Gamma, \mathbb{R}^{d,1})$**  By duality, the group of isometries of Minkowski space, that is  $O(d, 1) \times \mathbb{R}^{d,1}$ , acts on co-Minkowski space, preserving the degenerate metric (see Remark 2.2). For our purpose, it will be more relevant to restrict ourselves to the action of  $O_0(d, 1) \times \mathbb{R}^{d,1}$ , where  $O_0(d, 1)$  is the connected component of the identity of  $O(d, 1)$ . If  $\Gamma$  is a Kleinian cocompact subgroup of  $O_0(d, 1)$ , then the representations of  $\Gamma$  into  $O_0(d, 1) \times \mathbb{R}^{d,1}$  are parametrized by maps  $\tau : \Gamma \rightarrow \mathbb{R}^{d,1}$  satisfying a cocycle relation. Let  $Z^1(\Gamma, \mathbb{R}^{d,1})$  be the space of cocycles.

The choice of two totally geodesic embedding of  $\mathbb{H}^d$  (on which  $\Gamma$  acts) into co-Minkowski space will give different cocycles, related by a coboundary conditions. So we are interested in the space  $H^1(\Gamma, \mathbb{R}^{d,1})$ , the quotient of the space of cocycles by the coboundaries. From an extrinsic point of view, the vector space  $H^1(\Gamma, \mathbb{R}^{d,1})$  is the space of deformations of  $\Gamma$  into the group of affine isometries, up to conjugacy by translations. But  $H^1(\Gamma, \mathbb{R}^{d,1})$  encodes many much informations:

- due to Mostow rigidity theorem, for  $d > 2$ , it is not possible to non-trivially deform  $\Gamma$  among Kleinian subgroups of  $O(d, 1)$ . But it is possible to look at deformations of the canonical representation of  $\Gamma$  into  $O(d + 1, 1)$ , that corresponds to the deformation of the flat conformal structure of  $\mathbb{H}^d/\Gamma$ . At an infinitesimal level, the deformations are parametrized by  $H^1(\Gamma, \mathfrak{so}(d + 1, 1))$ . Due to the well-known splitting  $\mathfrak{so}(d + 1, 1) = \mathfrak{so}(d, 1) \oplus \mathbb{R}^{d,1}$ , we have that

$$H^1(\Gamma, \mathfrak{so}(d + 1, 1)) = H^1(\Gamma, \mathfrak{so}(d, 1)) \oplus H^1(\Gamma, \mathbb{R}^{d,1})$$

but due to the Calabi–Weil infinitesimal rigidity theorem  $H^1(\Gamma, \mathfrak{so}(d, 1))$  reduces to 0 [Kap09, 8.10].

On the other hand,  $H^1(\Gamma, \mathbb{R}^{d,1})$  is also isomorphic, as a linear space, to the tangent space of the Teichmüller space at (the conjugacy class of)  $\Gamma$ , when we consider the Teichmüller space as the space of discrete, faithful representations of  $\Gamma$  into the isometries of the hyperbolic plane up to conjugacy, see Section 3.4;

- there is a natural isomorphism between  $H^1(\Gamma, \mathbb{R}^{d,1})$  and the space of traceless symmetric Codazzi tensors on  $\mathbb{H}^d/\Gamma$  (see Proposition 3.17 for a proof using extrinsic co-Minkowski geometry), and the space of traceless symmetric Codazzi tensors parametrizes the space of infinitesimal deformations of the flat conformal structure of  $\mathbb{H}^d/\Gamma$ , as well as the space of infinitesimal deformations of the Riemannian metric of  $\mathbb{H}^d/\Gamma$  preserving the total volume and the harmonicity of the curvature [Laf83];
- $H^1(\Gamma, \mathbb{R}^{d,1})$  parametrizes the space of future complete flat globally hyperbolic maximal Cauchy compact spacetimes (in short, future complete flat GHMC spacetimes), with  $\Gamma$  as the linear part of the holonomy, see [Mes07, ABB<sup>+</sup>07, Bar05, Bon05] for more details and precise definitions. The universal covers of such spacetimes isometrically embed as convex sets in Minkowski space, whose duals in co-Minkowski space define the convex cores that will be mentioned below, see Remark 3.24.

As a consequence of the first point, for  $d = 2$ ,  $H^1(\Gamma, \mathbb{R}^{d,1})$  is a vector space of dimension  $(6g - 6)$ , where  $g$  is the genus of  $\mathbb{H}^2/\Gamma$ . For  $d > 3$ , it is not clear whether  $H^1(\Gamma, \mathbb{R}^{d,1})$  is trivial or not. A classical result is that it has dimension at least  $r$  if  $\mathbb{H}^d/\Gamma$  contains  $r$  disjoint embedded totally geodesic hypersurfaces [Laf83, Kou85, JM87]. We give an elementary co-Minkowski proof of this fact in Section 3.1. See for example [Apa90] and [JM87] for more informations, and [BS07] for up-to-date references about this question.

The action of  $\Gamma_\tau$ , that is  $\Gamma$  deformed by an element  $\tau$  of  $Z^1(\Gamma, \mathbb{R}^{d,1})$ , onto co-Minkowski space is also interesting in its own. Namely, here too, it is a baby toy model, this time comparing to the study of quasi-Fuchsian hyperbolic manifolds on the one hand, and to AdS GHMC manifolds on the other one (they are the Lorentzian analogues of quasi-Fuchsian hyperbolic manifolds). We will focus on the following aspects. Let  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ .

- There exists a smooth hypersurface invariant under the action of  $\Gamma_\tau$ . This is a simple illustration of the general “Ehresmann–Weil–Thurston principle”, see Proposition 3.13.
- The group  $\Gamma_\tau$  acts freely and properly discontinuously on co-Minkowski space, and the quotient gives a  $(d + 1)$ -dimensional manifold homeomorphic to  $\mathbb{H}^d/\Gamma \times \mathbb{R}$  (see Lemma 3.1).
- The co-Minkowski manifold  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$  has a *convex core*, i.e. it contains a non-empty compact convex set. So the action of  $\Gamma_\tau$  on co-Minkowski space is *convex cocompact* in the sense of [DGK17b, DGK17a].
- The co-Minkowski manifold  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$  contains a unique “mean” hypersurface, that is with vanishing mean curvature. This situation is reminiscent of *almost Fuchsian manifolds*, a particular case of quasi-Fuchsian manifolds which contain a unique minimal surface, see [KS07].
- Moreover,  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$  is foliated by CMC hypersurfaces, equidistant to the mean hypersurface, see Remark 3.16.

We consider that co-Minkowski space is a toy model, because with a pedestrian approach, we are able to give an almost self-contained presentation of the different properties evoked above.

**An asymmetric norm** Until this point, all the mentioned results were previously more or less known, at least under the form of dual statements in Minkowski space. Also, the present survey contains the following original contribution.

As we said, the quotient of co-Minkowski space by  $\Gamma_\tau$  has a convex core, and a unique mean hypersurface, contained in the convex core. The mean distance between the lower boundary component of the convex core and this mean hypersurface gives a non-negative number, which is uniquely defined by the class in  $H^1(\Gamma, \mathbb{R}^{d,1})$  of  $\tau$ . This gives a map from  $H^1(\Gamma, \mathbb{R}^{d,1})$  to  $\mathbb{R}_+$ , which is actually an asymmetric norm on  $H^1(\Gamma, \mathbb{R}^{d,1})$ , see Section 3.3.2. We will call it the  $S_1$  norm (see Remark 3.26 for the signification of  $S_1$ ).

The symmetrization of the  $S_1$  norm is:

- the volume of the convex core;<sup>3</sup>
- a “mean distance” between the future complete and the past complete flat GHMC having the same holonomy (see Remark 3.24).

In dimension 2, it appears that this asymmetric norm corresponds to the *earthquake norm* introduced by Thurston in [Thu98]. In particular, we obtain a new proof of Theorem 5.2 in [Thu98], saying that the earthquake norm is an asymmetric norm on the tangent of Teichmüller space. The tangent space of Teichmüller space can be identified with the space of measured geodesic laminations, and the earthquake norm is the total length of the lamination, see Section 3.4.

In turn, the volume of the convex core is the sum of the total length of the bending laminations of its boundary. Here again, this result should be compared with its more involved analogues in the hyperbolic and anti-de Sitter cases [Bro03, BST17].

Using two successive identifications of the tangent space of Teichmüller space with its cotangent space and a formula of Wolpert, the earthquake norm defines another asymmetric norm on the tangent space of Teichmüller space, the *length norm*, see (41) for a formula. The length norm defines an asymmetric Finsler structure on Teichmüller space, that in turn defines a distance, now called the *Thurston asymmetric distance*, and introduced by Thurston in [Thu98]. This distance recently attracted attention [PT07, PS15, Wal14]. Note that the earthquake norm also induces an asymmetric distance on Teichmüller space, but, to the best of our knowledge, nothing is known about this distance.

**Anosov feature** In the third and last part of the present survey, we see that co-Minkowski space is also a baby toy model for the theory of Anosov representations, which has known during the recent years, after the pioneering work of F. Labourie [Lab06] a series of development (see [GGKW17], [CLS17], [GW12], [BCLS15], [KLP16], see also [Bar16] for a complementary discussion on Anosov representations in the context of Lorentzian geometry, and [Gho17] for a proof of the Anosov character of the representations considered in the present survey).

Once more, it turns out that in the context of co-Minkowski space the theory of Anosov representations reduces to a particularly simple form. Moreover, this point of view provides a proof of the fact that convergence of cocycle implies *uniform* convergence of limit curves (Lemma 4.11).

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<sup>3</sup>This fact was noted to the first author by Andrea Seppi.

## 2 Co-Minkowski geometry

Co-Minkowski space is the space of (unoriented) spacelike hyperplanes of Minkowski space. We first investigate the space of oriented spacelike hyperplanes (Section 2.1). Then we introduce a cylindrical affine model for co-Minkowski space, similar to the Klein ball model of hyperbolic space (Section 2.2). In the cylindrical model, the co-Minkowski space is the cylinder  $B^d \times \mathbb{R}$ , where  $B^d$  is the open unit ball of  $\mathbb{R}^d$  centered at the origin. In particular, extrinsic co-Minkowski geometry of graphs of maps  $h : B^d \rightarrow \mathbb{R}$  can be investigated (Section 2.3).

### 2.1 Definition of co-Minkowski space

#### 2.1.1 Space of spacelike hyperplanes

Let us recall that the *Minkowski space*  $\mathbb{R}^{d,1}$  of Lorentzian geometry is the affine space  $\mathbb{R}^{d+1}$  endowed with the bilinear form

$$\langle x, y \rangle_{d,1} = x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1} .$$

A hyperplane  $P$  of  $\mathbb{R}^{d,1}$  is *spacelike* (resp. *timelike*, *lightlike*) if the restriction of  $\langle \cdot, \cdot \rangle_{d,1}$  to  $P$  is positive-definite (resp. has signature  $(+, \dots, +, -)$ , is degenerate). The isometry group of  $\mathbb{R}^{d,1}$  is  $O(d, 1) \times \mathbb{R}^{d,1}$ : it is made of translations and linear transformations preserving  $\langle \cdot, \cdot \rangle_{d,1}$ .

Linear spacelike hyperplanes are parametrized by the set of *future* unit normal vectors (for  $\langle \cdot, \cdot \rangle_{d,1}$ ):

$$\mathcal{H}^d := \{x \in \mathbb{R}^{d,1} \mid \langle x, x \rangle_{d,1} = -1, x_{d+1} > 0\} .$$

Let  $g_{\mathcal{H}^d}$  be the metric induced by  $\langle \cdot, \cdot \rangle_{d,1}$  on the tangent spaces of  $\mathcal{H}^d$ . It is well known that  $(\mathcal{H}^d, g_{\mathcal{H}^d})$  is a model of the  $d$ -dimensional hyperbolic space.

Let  $P$  be an affine spacelike hyperplane of  $\mathbb{R}^{d,1}$ . If  $n \in \mathcal{H}^d \subset \mathbb{R}^{d,1}$  is the future timelike unit normal to  $P$ , then there exists  $h \in \mathbb{R}$  such that

$$P = \{y \in \mathbb{R}^{d,1} \mid \langle y, n \rangle_{d,1} = h\} .$$

This defines a point

$$\tilde{P}^* = (n, h)$$

in  $\mathbb{R}^{d+1} \times \mathbb{R} = \mathbb{R}^{d+2}$ . More precisely, the point  $\tilde{P}^*$  belongs to one of the connected component of the degenerate quadric

$$\text{coMin}^{d+1} := \{x \in \mathbb{R}^{d+2} \mid \langle x, x \rangle_{d,1,0} = -1\}$$

where

$$\langle (x_1, \dots, x_{d+1}, x_t), (y_1, \dots, y_{d+1}, y_t) \rangle_{d,1,0} = x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1} .$$

Note that  $\text{coMin}^{d+1}$  is the space of oriented spacelike hyperplanes of Minkowski space. See Figure 2.

We will denote by  $g_{\text{coM}^{d+1}}$  the degenerate  $(0, 2)$ -tensor induced by  $\langle \cdot, \cdot \rangle_{d,1,0}$  on the tangents spaces of  $\text{coMin}^{d+1}$ . The connected component  $\text{coMin}_+^{d+1} = \text{coMin}^{d+1} \cap \{x_{d+1} > 0\}$  of  $\text{coMin}^{d+1}$  containing the point  $\tilde{P}^*$  is homeomorphic to  $\mathcal{H}^d \times \mathbb{R}$ . In those coordinates, the degenerate metric  $g_{\text{coM}^{d+1}}$  on  $\text{coMin}_+^{d+1}$  writes as

$$g_{\text{coM}^{d+1}} = g_{\mathcal{H}^d} + 0 dx_t .$$

We also introduce the fibration:

$$\pi : \text{coMin}_+^{d+1} \rightarrow \mathcal{H}^d$$

mapping  $(x_1, \dots, x_{d+1}, x_t)$  to  $(x_1, \dots, x_{d+1})$ . It is a principal  $\mathbb{R}$ -bundle; it is an isometry, and the fibers are precisely tangent to the kernel of the degenerate metric  $g_{\text{coM}^{d+1}}$ .

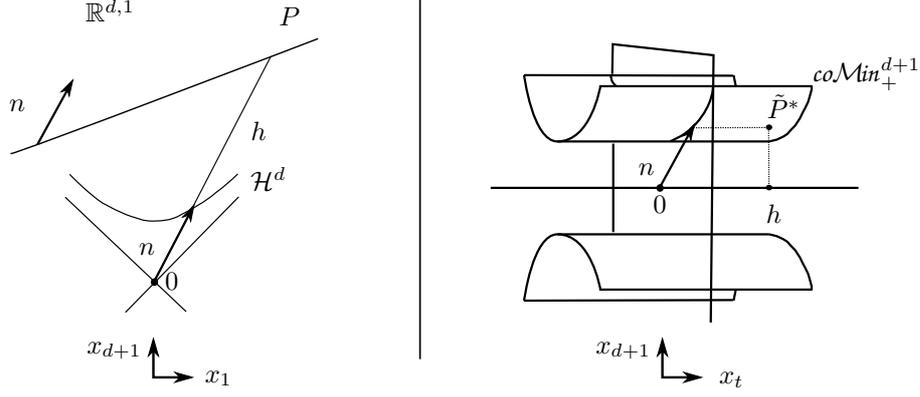


Figure 2: The dual of a spacelike hyperplane of Minkowski space  $\mathbb{R}^{d,1}$  in  $coMin^{d+1}$ . In the picture,  $d = 1$ .

### 2.1.2 Isometries

As the “metric”  $g_{coM^{d+1}}$  is degenerate, it will be more relevant to consider a group acting on  $coMin^{d+1}$ . As an isometry of Minkowski space sends spacelike hyperplanes onto spacelike hyperplanes, it acts naturally on  $coMin^{d+1}$ . This is the way we define the isometry group of  $coMin^{d+1}$ . More precisely, it is immediate that if

$$P = \{y \in \mathbb{R}^{d,1} \mid \langle y, n \rangle_{d,1} = h\}$$

is a spacelike hyperplane of  $\mathbb{R}^{d,1}$  and  $A \in O(d, 1)$ , so that  $\tilde{P}^* = (n, h)$ , then

$$\widetilde{AP}^* = (An, h)$$

and if  $v \in \mathbb{R}^{d,1}$ ,

$$\widetilde{P + v}^* = (n, \langle v, n \rangle_{d,1} + h).$$

So  $O(d, 1) \times \mathbb{R}^{d,1}$  acts linearly on  $coMin^{d+1}$  via the representation

$$(A, v) \mapsto \left( \begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline {}^t v J A & 1 \end{array} \right), \quad (1)$$

where  $J = \text{diag}(1, \dots, 1, -1)$  (recall that  $A \in O(d, 1)$  if and only if  ${}^t A = J A^{-1} J$ ). So we define the *isometry group* of  $coMin^{d+1}$  as  $O(d, 1) \times \mathbb{R}^{d,1}$  with the action on  $\mathbb{R}^{d+2}$  induced by the representation (1). In particular, the group structure on  $O(d, 1) \times \mathbb{R}^{d,1}$  is

$$(A_1, v_1) \cdot (A_2, v_2) = (A_1 A_2, v_1 + A_1 v_2). \quad (2)$$

**Remark 2.1.** Let  $O_+(d, 1)$  be the subgroup of  $O(d, 1)$  preserving  $\mathcal{H}^d$ . Then,  $O_+(d, 1) \times \mathbb{R}^{d,1}$  preserves the connected component  $coMin_+^{d+1}$ , and the fibration  $\pi$  is  $O_+(d, 1) \times \mathbb{R}^{d,1}$ -equivariant. The elements of  $O_+(d, 1) \times \mathbb{R}^{d,1}$  inducing the identity map on  $\mathcal{H}^d$  are precisely the translations (elements of  $\mathbb{R}^{d,1}$ ).

Every fiber of  $\pi$  admits a natural affine structure, for which they are individually isomorphic to the real line. The action of  $O_+(d, 1) \times \mathbb{R}^{d,1}$  preserves this affine structure along the fibers.

**Remark 2.2.** The isometry group of  $coMin^{d+1}$  is smaller than the group of transformations preserving the degenerate metric  $g_{coM^{d+1}}$ . For example, for  $c > 0$ , the map  $H_c : \mathbb{R}^{d+2} \rightarrow \mathbb{R}^{d+2}$ ,  $H_c(x) = (x_1, x_2, \dots, x_{d+1}, cx_t)$ , preserves  $\langle \cdot, \cdot \rangle_{d,1,0}$  (hence it preserves  $coMin^{d+1}$  and  $g_{coM^{d+1}}$ ), but by definition it is not an isometry of  $coMin^{d+1}$ .

There does not exist any (non-degenerate) semi-Riemannian metric on  $coMin^{d+1}$  invariant under the isometry group of  $coMin^{d+1}$  [FS18, Fact 2.27].

### 2.1.3 Connection, geodesics

We have now the hypersurface  $coMin^{d+1}$  in  $\mathbb{R}^{d+2}$  together with an “isometry group” and a degenerate metric  $g_{coM^{d+1}}$ . As those elements are coming from the degenerate form  $\langle \cdot, \cdot \rangle_{d,1,0}$  on the ambient  $\mathbb{R}^{d+2}$ , there is no obvious metric notion of “unit normal vector” to  $coMin^{d+1}$ . Nevertheless, we can proceed similarly to classical affine differential geometry [NS94]. Namely, at a point  $x \in coMin^{d+1}$ , let us define as a “normal field” the vector field  $N(x) = x$ . Obviously,  $N$  is transverse to  $coMin^{d+1}$  and invariant under the group of isometries of  $coMin^{d+1}$ . The choice of this normal field allows to define a connection  $\nabla^{coM^{d+1}}$  on  $coMin^{d+1}$  induced by the canonical connection  $D$  of the ambient linear space  $\mathbb{R}^{d+2}$ :

$$D_Y X = \nabla_Y^{coM^{d+1}} X + \langle X, Y \rangle_{d,1,0} N .$$

The following facts are easily checked, see [FS18, Section 4.2].

**Fact 2.3.** *The connection  $\nabla^{coM^{d+1}}$  has the following properties:*

- *it is torsion free,*
- *compatible with the degenerate metric  $g_{coM^{d+1}}$ ,*
- *invariant under isometries,*
- *its (unparameterized) geodesics are intersection of  $coMin^{d+1}$  with linear planes of  $\mathbb{R}^{d+2}$ .*

It follows from the last point that the intersection of  $coMin^{d+1}$  with linear  $k$ -planes of  $\mathbb{R}^{d+2}$  are totally geodesic. Those intersections will play a fundamental role as the following fact shows.

**Fact 2.4.** *The intersection of  $coMin^{d+1}$  with a linear  $k$ -planes of  $\mathbb{R}^{d+2}$  is isometric (for the metric induced by  $g_{coM^{d+1}}$ ) to the hyperbolic space of dimension  $k$ .*

*Moreover,  $\nabla^{coM^{d+1}}$  coincides with the Levi-Civita connection of the hyperbolic metric on any such subspace.*

*Proof.* Immediate as one can always find an isometry of  $coMin^{d+1}$  sending a linear  $k$ -plane to a linear  $k$ -plane contained in  $\{x_t = 0\}$ .  $\square$

### 2.1.4 Co-Minkowski space

The co-Minkowski space is the space of unoriented spacelike hyperplanes of Minkowski space, that is the quotient of  $coMin^{d+1}$  by the antipodal map.

**Definition 2.5.** *The co-Minkowski space  $*\mathbb{R}^{d,1}$  is the following subspace of the projective space:  $*\mathbb{R}^{d,1} = coMin^{d+1}/\{\pm Id\}$ , endowed with the push-forward of the degenerate metric  $g_{coM^{d+1}}$ , denoted by  $g_{*\mathbb{R}^{d,1}}$ .*

The connection  $\nabla^{coM^{d+1}}$  also induces a connection  $\nabla^{*\mathbb{R}^{d,1}}$  on  $*\mathbb{R}^{d,1}$ .

We define the *isometry group* of  $*\mathbb{R}^{d,1}$  as the image of  $O(d,1) \times \mathbb{R}^{d,1}$  into  $PGL(d+2)$ , by a projective quotient of the representation given by (1).

The map  $\pi : coMin_+^{d+1} \rightarrow \mathcal{H}^d$  induces a  $\mathbb{R}$ -fibration  $*\pi : *\mathbb{R}^{d,1} \rightarrow \mathcal{H}^d$ , which is an isometry, and  $O(d,1) \times \mathbb{R}^{d,1}$ -equivariant.

It will be interesting to work in a particular affine model of co-Minkowski space. This will be the cylindrical coordinates introduced in the next section.

## 2.2 Cylindrical model

### 2.2.1 Klein ball model of the hyperbolic space

We have seen that the subspace  $\mathcal{H}^d$  of Minkowski space, endowed with the induced metric, is a model of the hyperbolic space. It is isometric to the subset  $\{x \in \mathbb{R}^{d,1} | \langle x, x \rangle_{d,1} < 0\}$  of the projective space  $\mathbb{P}(\mathbb{R}^{d,1})$  endowed with the push-forward metric.

The *Klein ball model* of the hyperbolic space is the image of the projective model of the hyperbolic space in the affine chart  $\{x_{d+1} = 1\}$ . As a set, it is the open Euclidean unit ball  $B^d$ . The push-forward of the hyperbolic metric on  $B^d$  is denoted by  $g_{\mathbb{H}^d}$ . We will sometimes

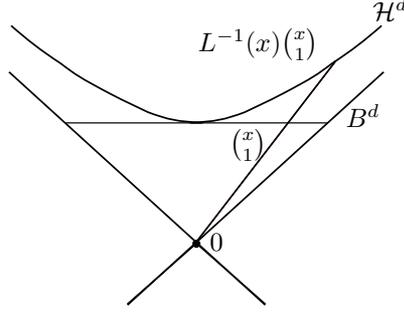


Figure 3: The hyperboloid  $\mathcal{H}^d$  and the Klein ball model of the hyperbolic space.

use the notation  $\mathbb{H}^d$  to designate the hyperbolic space  $(B^d, g_{\mathbb{H}^d})$ . In the remainder of this section, we give explicit formulas relating the hyperbolic geometry on  $B^d$  to the standard Euclidean geometry on  $B^d$ , that will be needed in the sequel of the paper.

If  $x \in B^d$ , then the vector  $\begin{pmatrix} x \\ 1 \end{pmatrix}$  of  $\mathbb{R}^{d,1}$  rescaled by the factor  $L^{-1}(x)$  belongs to  $\mathcal{H}^d$ , where

$$L(x) = \sqrt{1 - \|x\|^2}$$

and  $\|\cdot\|$  is the Euclidean norm on  $B^d$ :

$$\|(x_1, \dots, x_d)\| = \sqrt{x_1^2 + \dots + x_d^2},$$

see Figure 3.

The expression of the hyperbolic metric  $g_{\mathbb{H}^d}$  in the Klein ball model is:

$$g_{\mathbb{H}^d}(x)(X, Y) = L(x)^{-2} \langle X, Y \rangle_d + L(x)^{-4} \langle x, X \rangle_d \langle x, Y \rangle_d. \quad (3)$$

where  $\langle \cdot, \cdot \rangle_d$  is the standard Euclidean metric on  $\mathbb{R}^d \supset B^d$ ,  $x \in B^d$ ,  $X, Y \in T_x B^d \cong \mathbb{R}^d$ . In order to help computations, one may note that

$$D_X L^{-1}(x) = L^{-3}(x) \langle x, X \rangle_d \quad (4)$$

and

$$\text{Hess } L = -L g_{\mathbb{H}^d}, \quad (5)$$

where Hess is the usual Hessian on  $\mathbb{R}^d$ .

If  $\omega_{B^d}$  is the restriction to  $B^d$  of the Euclidean volume form, and  $\omega_{\mathbb{H}^d}$  is the volume form on  $B^d$  associated to the hyperbolic metric  $g_{\mathbb{H}^d}$ , from (3) one obtains

$$\omega_{B^d} = L^{d+1} \omega_{\mathbb{H}^d}. \quad (6)$$

The main feature of the Klein ball model of the hyperbolic space is that the (unparameterized) geodesics of  $g_{\mathbb{H}^d}$  are exactly the affine segments in  $B^d$ . This is straightforward, as the geodesics of  $\mathcal{H}^d$  are the intersections of  $\mathcal{H}^d$  with linear timelike planes of  $\mathbb{R}^{d,1}$ . This gives the following correspondence between the connections, see [FS18, Lemma 4.17].

**Proposition 2.6** (Weyl formula). *If  $\nabla^{\mathbb{H}^d}$  is the Levi-Civita connection of  $g_{\mathbb{H}^d}$  and  $D$  is the canonical connection on  $B^d$ , then*

$$\nabla_X^{\mathbb{H}^d} Y = D_X Y + L^{-2}(x) (\langle x, X \rangle_d Y + \langle x, Y \rangle_d X). \quad (7)$$

**Corollary 2.7.** *If Hess $^{\mathbb{H}^2}$  is the Hessian given by  $\nabla^{\mathbb{H}^2}$ , then, for a smooth map  $f : B^d \rightarrow \mathbb{R}$ ,*

$$\text{Hess}^{\mathbb{H}^2} f(x)(X, Y) = \text{Hess } f(x)(X, Y) - L^{-2}(x) (\langle x, X \rangle_d \text{d} f(x)(Y) + \langle x, Y \rangle_d \text{d} f(x)(X)). \quad (8)$$

Also,

$$L^{-1}(x) \text{Hess } f(x)(X, Y) = \left( \text{Hess}^{\mathbb{H}^2} (L^{-1} f)(x)(X, Y) - (L^{-1} f)(x) g_{\mathbb{H}^d}(x)(X, Y) \right). \quad (9)$$

*Proof.* (8) follows from (7) and

$$\text{Hess}^{\mathbb{H}^d} f(x)(X, Y) = X.Y.f(x) - d f(x)(\nabla_X^{\mathbb{H}^d} Y) . \quad (10)$$

Finally, (9) comes from (8), (3) and

$$\text{Hess } fg = f\text{Hess } g + g\text{Hess } f + d f \otimes d g + d g \otimes d f . \quad (11)$$

□

**Fact 2.8.** *If  $\Delta$  is the Euclidean Laplacian on  $B^d$ , then*

$$\text{tr}_{g_{\mathbb{H}^d}} \text{Hess } f(x) = L^2(x)(\Delta f - \text{Hess } f(x)(x, x)) . \quad (12)$$

*If  $\Delta^{\mathbb{H}^d}$  is the Laplacian on  $B^d$  given by  $g_{\mathbb{H}^d}$ , then*

$$\text{tr}_{g_{\mathbb{H}^d}} L^{-1} \text{Hess } f = \Delta^{\mathbb{H}^d} (L^{-1} f) - d(L^{-1} f) . \quad (13)$$

*Proof.* Let  $A$  be the linear operator such that  $\text{Hess } f(x)(X, Y) = g_{\mathbb{H}^d}(x)(AX, Y)$ . For  $x \neq 0$ , let  $(e_i)_{1, \dots, d}$  be an orthonormal Euclidean basis of  $T_x B^d$ , such that  $e_1 = x/\|x\|$ . The definition of  $A$  and (3) give, for  $i > 1$ ,

$$\langle Ae_i, e_i \rangle_d = L^2(x)g_{\mathbb{H}^d}(x)(Ae_i, e_i) = L^2(x) \text{Hess } f(x)(e_i, e_i) ,$$

and

$$\langle Ae_1, e_1 \rangle_d = L^2(x) \text{Hess } h(x)(e_1, e_1) + L^{-2}(x)\langle x, Ax \rangle_d .$$

Also from the definition of  $A$  and (3),

$$L^{-2}(x)\langle x, Ax \rangle_d = L^2(x)g_{\mathbb{H}^d}(x)(x, Ax) = L^2(x) \text{Hess } f(x)(x, x) .$$

(12) follows from  $\text{tr}_{g_{\mathbb{H}^d}} \text{Hess } f(x) = \sum_{i=1}^d \langle Ae_i, e_i \rangle_d$ . Also, (13) is immediate from (9). □

Let us end this section with some basic facts about (smooth) hyperbolic Codazzi tensors.

**Definition 2.9.** *A  $(0, 2)$ -tensor  $C$  on  $\mathbb{H}^d$  is a (hyperbolic) Codazzi tensor if it satisfies the the Codazzi equation on  $\mathbb{H}^d$ :*

$$(\nabla_X^{\mathbb{H}^d})C(Y, Z) = (\nabla_Y^{\mathbb{H}^d})C(X, Z) .$$

**Lemma 2.10.** *Let  $C$  be a  $(0, 2)$ -tensor on  $B^d$ . Then  $C$  is a hyperbolic Codazzi tensor if and only if*

$$D_X(LC)(Y, Z) = D_Y(LC)(X, Z) .$$

*Proof.* The definition of Codazzi tensor means that

$$X.C(Y, Z) - C(\nabla_X^{\mathbb{H}^d} Y, Z) - C(Y, \nabla_X^{\mathbb{H}^d} Z) = Y.C(X, Z) - C(\nabla_Y^{\mathbb{H}^d} X, Z) - C(X, \nabla_Y^{\mathbb{H}^d} Z) .$$

Developing this expression using (7), one obtains, at a point  $x$ ,

$$D_X C(x)(Y, Z) - L^{-2}(x)\langle x, X \rangle_d C(Y, Z) = D_Y C(x)(X, Z) - L^{-2}(x)\langle x, Y \rangle_d C(X, Z) .$$

Writing  $C = L^{-1}LC$ , developing the above expression and using (4) leads to the result. □

**Fact 2.11.** *Let  $S$  be a  $(0, 2)$ -tensor on  $B^d$ . If  $D_X S(Y, Z) = D_Y S(X, Z)$ , then there exists a function  $F = (F_1, \dots, F_n)$  with  $F_i : B^d \rightarrow \mathbb{R}$  such that  $S$  is the Jacobian matrix of  $F$ .*

*Proof.* Let  $\Omega_j = \sum_{i=1}^d S_{ij} dx^i$ . As  $\frac{\partial S_{ij}}{\partial x_k} = \frac{\partial S_{kj}}{\partial x_i}$ ,  $d\Omega_j = 0$ , so by Poincaré Lemma, there exists a function  $F_j : B^d \rightarrow \mathbb{R}$  such that  $dF_j = \Omega_j$ . □

**Fact 2.12.** *Let  $F = (F_1, \dots, F_d)$  with  $F_j : B^d \rightarrow \mathbb{R}$ . Then there exists  $f : B^d \rightarrow \mathbb{R}$  with  $\frac{\partial f}{\partial x_i} = F_i$  if and only if  $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ .*

*In other term, the Jacobian matrix of  $F$  is a Hessian matrix (namely the one of  $f$ ) if and only if it is a symmetric matrix.*

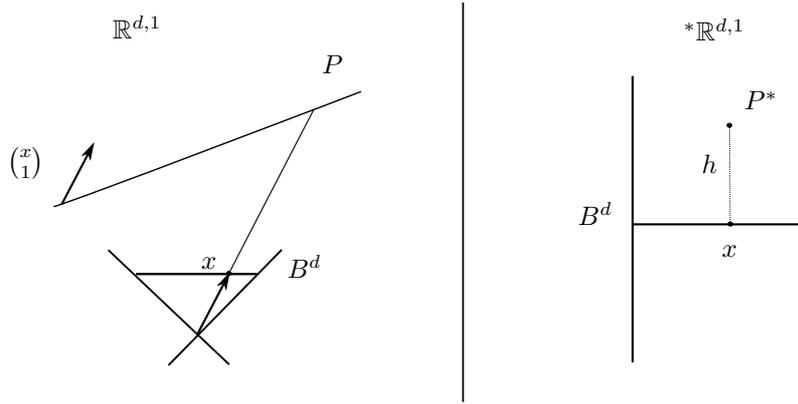


Figure 4: The dual  $P^*$  of the hyperplane  $P = \{y \in \mathbb{R}^{d,1} \mid \langle \binom{x}{1}, y \rangle_{d,1} = h\}$ .

*Proof.* One implication is Schwarz's theorem. On the other direction, the one-form  $\omega = \sum_{i=1}^d F_i dx^i$  is closed by hypothesis, hence exact by Poincaré Lemma, and it suffices to set  $\omega = df$ .  $\square$

We finally obtain the following classical result [BBD<sup>+</sup>81, OS83, BS16].

**Lemma 2.13.** *Let  $C$  be a  $(0,2)$ -tensor on  $B^d$ . Then  $C$  is a symmetric hyperbolic Codazzi tensor if and only if there exists  $f : B^d \rightarrow \mathbb{R}$  such that*

$$C = L^{-1} \text{Hess } f .$$

### 2.2.2 Affine representation of co-Minkowski space

To keep track of some relevant affine notions such as convexity, we will work in an affine model of co-Minkowski space. Namely, we will consider the affine model of co-Minkowski space given by the central projection of  $\text{coMin}_+^{d+1}$  onto the hyperplane  $\{x_{d+1} = 1\}$  of  $\mathbb{R}^{d+2}$ . Observe that in doing so, we favor the coordinate  $x_{d+1}$ , i.e. we distinguish the future timelike vector  $(0, \dots, 0, 1)$  of  $\mathbb{R}^{d,1}$ . We will go back on this remark in Section 4. In the hyperplane  $\{x_{d+1} = 1\}$ , the image of  $\text{coMin}_+^{d+1}$  is the cylinder  $B^d \times \mathbb{R}$ , where  $B^d$  is the open unit ball centered at the origin of  $\mathbb{R}^d$ .

We denote by  $\pi : B^d \times \mathbb{R} \rightarrow B^d$  the projection on the first factor. It corresponds to the fibration  $\pi : \text{coMin}_+^{d+1} \rightarrow \mathcal{H}^d$ . We will call *vertical lines* the fibers of  $\pi$ . They correspond to parallel spacelike hyperplanes in Minkowski space.

**Remark 2.14.** In those coordinates  $B^d \times \mathbb{R} \subset \mathbb{R}^{d+1}$ , the degenerate metric  $g_{*\mathbb{R}^{d,1}}$  of co-Minkowski space is  $g_{\mathbb{H}^d} + 0 dx_t^2$ . The degenerate metric  $g_{*\mathbb{R}^{d,1}}$  defines a "distance" between points of co-Minkowski space. Actually this distance is nothing but the Klein projective metric: if  $x, y \in B^d \times \mathbb{R}$ , then they are on a line meeting  $\partial B^d \times \mathbb{R} \cup \{\infty\}$  either at two distinct points  $I, J$ , or at  $I = J = \infty$ . Then the Klein projective distance is  $d(x, y) = \frac{1}{2} |\ln[x, y, I, J]|$ , where  $[\cdot, \cdot, \cdot, \cdot]$  is the cross-ratio, see [FS18].

**Remark 2.15.** The *boundary at infinity* of co-Minkowski space is  $\partial B^d \times \mathbb{R}$ . It parametrizes the set of lightlike affine hyperplanes of Minkowski space, and it is called *Penrose boundary* in [Bar05]. Note that  $(\mathbb{R}^d \setminus \bar{B}^d) \times \mathbb{R}$  parametrizes the set of affine timelike hyperplanes of Minkowski space, but we don't need to consider it.

The interest of an affine model is essentially given by the following facts. The first one is an immediate consequence of the last point of Fact 2.3.

**Fact 2.16.** *(Unparameterized) geodesics of  $*\mathbb{R}^{d,1}$  in the cylindrical model  $B^d \times \mathbb{R}$  are (affine) geodesic segments.*

The second fact follows from Fact 2.4 and by construction.

**Fact 2.17.** *The intersection of  $B^d \times \mathbb{R}$  with any affine  $k$ -plane not containing a vertical line, with the metric induced by  $g_{*\mathbb{R}^{d,1}}$ , is isometric to the hyperbolic space of dimension  $k$ .*

*In particular,  $B^d \times \{0\} \cong B^d$  is the Klein ball model of the  $d$ -dimensional hyperbolic space.*

When  $k = d$ , we will call the intersection of  $B^d \times \mathbb{R}$  with a  $d$ -plane not containing a vertical line a *hyperbolic hyperplane*.

**Remark 2.18.** As every non-degenerate tangent plane of co-Minkowski space is isometric to the tangent plane of a hyperbolic space, the sectional curvature of co-Minkowski space is  $-1$ .

### 2.2.3 Duality

This cylindrical affine model can be directly described from Minkowski space as follows. Let  $P$  be an affine spacelike hyperplane of  $\mathbb{R}^{d,1}$ , and let  $(x, 1)$  be a normal vector, with  $x \in B^d$ . Then there exists a number  $h$  such that

$$P = \{y \in \mathbb{R}^{d,1} \mid \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, y \rangle_{d,1} = h\}$$

and  $P$  defines a point  $P^* = (x, h) \in B^d \times \mathbb{R}$ , see Figure 4.

Let us give more precisions about the ‘‘duality’’ between Minkowski space and co-Minkowski space. We already know that if  $P$  is a spacelike hyperplane of Minkowski space, then  $P^*$  is a point in  ${}^*\mathbb{R}^{d,1}$ . Conversely, if  $P$  is a hyperbolic hyperplane of  ${}^*\mathbb{R}^{d,1}$ , let  $P^*$  be the intersection of all the hyperplanes of Minkowski space whose duals are points in  $P$ . For future reference, let us express this fact in terms of the cylindrical coordinates  $B^d \times \mathbb{R}$ .

**Fact 2.19.** *Let  $P$  be a hyperbolic hyperplane of co-Minkowski space, which is the graph of the affine function  $h : B^d \rightarrow \mathbb{R}$ ,  $h(x) = \langle x, v \rangle_d + c$ . Then the point  $P^*$  dual to  $P$  has coordinates  $P^* = (v, -c) \in \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{d,1}$ .*

*In other terms, if  $P$  is a point of Minkowski space, then the hyperplane  $P^*$  in co-Minkowski space is the graph of the affine map  $h : B^d \rightarrow \mathbb{R}$ ,  $h(x) = \langle P, \begin{pmatrix} x \\ 1 \end{pmatrix} \rangle_{d,1}$ .*

*Proof.* Let us fix  $x \in B^d$ . Then the point  $X = (x, \langle v, x \rangle_d + c) \in B^d \times \mathbb{R}$  of co-Minkowski space belongs to  $P$ . Its dual is the spacelike hyperplane of Minkowski space defined as

$$X^* = \{(y, y_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mid \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} \rangle_{d,1} = \langle v, x \rangle_d + c\}$$

i.e.  $X^* = \{(y, y_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \mid \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} \rangle_{d,1} = \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ -c \end{pmatrix} \rangle_{d,1}\}$  and obviously  $(v, -c)$  belongs to this hyperplane. As  $x$  was arbitrary,  $(v, -c)$  belongs to all the hyperplanes dual to the points of  $P$ , that is the definition of  $P^*$ . □

The proof of the following facts are left to the reader.

**Fact 2.20.** 1. *If  $P$  is a hyperbolic hyperplane in co-Minkowski space  ${}^*\mathbb{R}^{d,1}$ , then  $P^*$  is a point in Minkowski space  $\mathbb{R}^{d,1}$  and  $(P^*)^* = P$ .*

2. *Let  $P$  and  $Q$  be two hyperbolic hyperplanes in  ${}^*\mathbb{R}^{d,1}$ .*

- (a) *if  $P$  and  $Q$  meet in  ${}^*\mathbb{R}^{d,1}$  then  $P^*$  and  $Q^*$  are joined by a spacelike segment in  $\mathbb{R}^{d,1}$ .*
- (b) *if  $P$  is strictly above  $Q$  in  $\bar{B}^d \times \mathbb{R}$ , then  $Q^* - P^*$  is a future directed timelike segment in  $\mathbb{R}^{d,1}$ .*
- (c) *if  $P$  and  $Q$  have a common point in  $\partial B^d \times \mathbb{R}$ , then  $P^*$  and  $Q^*$  are joined by a lightlike segment.*

The vector space structure of Minkowski space corresponds via duality to the vector space structure on the space of restrictions to  $B^d$  of affine maps.

**Fact 2.21.** Let  $h_Q$  and  $h_P$  be the restriction to  $B^d$  of affine maps, such that their graphs are the hyperbolic hyperplanes  $P, Q$  of co-Minkowski space, and let  $\lambda \in \mathbb{R}$ . Then the graph of  $h_P + \lambda h_Q$  is dual to the point  $P^* + \lambda Q^*$  of Minkowski space.

**Remark 2.22.** A convex spacelike hypersurface  $S$  of Minkowski space is the boundary of the intersection of half-spaces bounded by spacelike hyperplanes. A hypersurface is *F-convex* if it is the boundary of a spacelike convex hypersurface such that any spacelike vector hyperplane is the direction of a support plane, and if the surface is in the future side of its support planes. Each support plane  $P$  has a normal vector of the form  $\begin{pmatrix} x \\ 1 \end{pmatrix}$  for  $x \in B^d$ , so there is  $h(x) \in \mathbb{R}$  such that

$$P = \{y | \langle y, \begin{pmatrix} x \\ 1 \end{pmatrix} \rangle_{d,1} = h(x)\} .$$

The graph  $S^*$  of the function  $h$  in  $B^d \times \mathbb{R}$  is actually a convex hypersurface, see [FV16, BF17]. In more classical terms,  $h$  is the *support function* of the convex set  $K$  bounded by  $S$ :

$$h(x) = \max_{k \in K} \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, k \rangle_{d,1} . \quad (14)$$

Let us suppose furthermore that  $S$  is the graph of a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . Then if  $k \in K$  there is  $y \in \mathbb{R}^d$  such that  $k = \begin{pmatrix} y \\ f(y) \end{pmatrix}$ , and from (14),

$$h(x) = \max_{y \in \mathbb{R}^d} \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ f(y) \end{pmatrix} \rangle_{d,1} = \max_{y \in \mathbb{R}^d} \{ \langle x, y \rangle_d - f(y) \} ,$$

i.e.  $h$  is nothing but the conjugate (Legendre–Fenchel dual) of  $f$ .

In the same way, convex hypersurfaces of Minkowski space which are in the past side of their support planes have dual hypersurfaces in the cylindrical model of co-Minkowski space, which are graphs of concave function  $h : B^d \rightarrow \mathbb{R}$ .

**Example 2.23.** The dual surface of the hyperboloid  $\{y | \langle y, y \rangle_{d,1} = -t^2, y_{d+1} > 0\}$  is the graph of the function  $B^d \rightarrow \mathbb{R}$ ,  $x \mapsto -tL(x)$ . Note that this function is convex (see (5)). In the same way, dual surface of the hyperboloid  $\{y | \langle y, y \rangle_{d,1} = -t^2, y_{d+1} < 0\}$  is the graph of the concave function  $h(x) = tL(x)$ .

**Remark 2.24.** Any hypersurface in Minkowski space which is an envelope of spacelike hyperplanes has a dual hypersurface in co-Minkowski space. This is more easily seen in the other way. For any  $C^2$  function  $h : B^d \rightarrow \mathbb{R}$ , there exists a map  $\chi : B^d \rightarrow \mathbb{R}^{d,1}$ , the *normal representation*, such that  $P = \{y | \langle y, \begin{pmatrix} x \\ 1 \end{pmatrix} \rangle_{d,1} = h(x)\}$  is tangent to  $\chi(B^d)$  at the point  $\chi(x)$ , see [FV16, 2.12]. Pay attention to the fact that  $\chi$  is in general not a regular map, and that the concept of tangent hyperplane has to be understood in a generalized sense. The simplest example is when  $h$  is the restriction to  $B^d$  of an affine map: its graph is a hyperplane  $P$  in the cylindrical model  $B^d \times \mathbb{R}$  of co-Minkowski space, and  $\chi(B^d)$  is reduced to a point, the dual point of  $P$  in Minkowski space.

**Remark 2.25.** The duality between  $\mathbb{R}^{d,1}$  and  ${}^*\mathbb{R}^{d,1}$  can also be seen in  $\mathbb{R}^{d+2}$ , looking at  $\mathbb{R}^{d,1}$  as a degenerate quadric in  $\mathbb{R}^{d+2}$ . See [FS18, Section 2.5] for more details.

#### 2.2.4 Isometries in cylindrical coordinates

Let us write the action of the isometry group of co-Minkowski space in the cylindrical coordinates  $B^d \times \mathbb{R}$ . First let us state some facts about the action of hyperbolic isometries on  $B^d$ . The group  $O_+(d, 1)$  acts by isometries on the hyperbolic space  $\mathcal{H}^d$ , and hence on the Klein ball model. More precisely, let  $x \in B^d$  and  $A \in O_+(d, 1)$ . We will denote by  $A \cdot x$  the image of  $x$  by the isometry of the Klein ball model defined by  $A$ . We have

$$\frac{1}{(A \begin{pmatrix} x \\ 1 \end{pmatrix})_{d+1}} A \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} A \cdot x \\ 1 \end{pmatrix} . \quad (15)$$

Note that as  $A$  is a linear isometry of Minkowski space  $\mathbb{R}^{d,1}$ , we have

$$|(A \begin{pmatrix} x \\ 1 \end{pmatrix})_{d+1}|^2 (\|A \cdot x\|^2 - 1) = \|x\|^2 - 1$$

i.e.

$$(A \begin{pmatrix} x \\ 1 \end{pmatrix})_{d+1} = \frac{L(x)}{L(A \cdot x)}, \quad (16)$$

so, together with (15), one obtains

$$A \begin{pmatrix} x \\ 1 \end{pmatrix} = \frac{L(x)}{L(A \cdot x)} \begin{pmatrix} A \cdot x \\ 1 \end{pmatrix}. \quad (17)$$

For simplicity, let us fix also the following coordinate system; every element  $(x_1, \dots, x_{d+1})$  of  $\mathbb{R}^{d+1}$  has a horizontal component  $\bar{x} = (x_1, \dots, x_d)$  and a vertical component  $x_{d+1}$ . If  $\langle \bar{x}, \bar{y} \rangle_d$  is the scalar product of horizontal elements, we have, for  $x, y \in \mathbb{R}^{d,1}$ ,  $\langle x, y \rangle_{d,1} = \langle \bar{x}, \bar{y} \rangle_d - x_{d+1}y_{d+1}$ .

**Lemma 2.26.** *Let  $(x, h) \in B^d \times \mathbb{R}$  and  $(A, v) \in O_+(d, 1) \ltimes \mathbb{R}^{d,1}$ . Then the isometry of co-Minkowski space defined by  $(A, v)$  acts on the cylindrical coordinates as follows:*

$$(A, v)(x, h) = \left( A \cdot x, \frac{L(A \cdot x)}{L(x)} h + \langle A \cdot x, \bar{v} \rangle_d - v_{d+1} \right). \quad (18)$$

*Proof.* When the isometry is linear, i.e. when  $v = 0$ , the elements of the image of  $(x, h)$  by  $(A, v)$  are elements of  $\mathbb{R}^{d,1}$  satisfying:

$$\begin{aligned} h &= \left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, A^{-1} \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} \right\rangle_{d,1} \\ &= \left\langle A \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} \right\rangle_{d,1} \\ &\stackrel{(17)}{=} \left\langle \frac{L(x)}{L(A \cdot x)} \begin{pmatrix} A \cdot x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} \right\rangle_{d,1}. \end{aligned}$$

Therefore, the image of  $(x, h)$  by  $(A, 0)$  is  $(A \cdot x, \frac{L(A \cdot x)}{L(x)} h)$ .

In the case of a translation by a vector  $v = \begin{pmatrix} \bar{v} \\ v_{d+1} \end{pmatrix}$  we have:

$$\begin{aligned} h &= \left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} - \begin{pmatrix} \bar{v} \\ v_{d+1} \end{pmatrix} \right\rangle_{d,1} \\ &= \left\langle \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ y_{d+1} \end{pmatrix} \right\rangle_{d,1} - \langle x, \bar{v} \rangle_d + v_{d+1}. \end{aligned}$$

Hence the image of  $(x, h)$  by the translation is

$$(x, h + \langle x, \bar{v} \rangle_d - v_{d+1}).$$

The Lemma follows because from (2),  $(A, v) = (\text{Id}, v)(A, 0)$ . □

**Remark 2.27.** There is an easy way to see the action of  $O_+(d, 1)$  in the coordinates  $B^d \times \mathbb{R}$ . Actually,  $B^d \times \mathbb{R}$  is foliated by the graphs of the functions  $tL$ ,  $t \in \mathbb{R}$ . Note that those graphs are, for  $t \neq 0$ , the duals of the two-sheeted hyperboloids centered at the origin in Minkowski space, see Example 2.23. Observe that for the sheet with positive (respectively negative)  $x_{d+1}$ , the parameter  $t$  is negative (respectively positive). Hence if  $(x, h) \in B^d \times \mathbb{R}$  belongs to the graph of  $tL$  for some  $t$ , then for any  $A \in O_+(d, 1)$ ,  $(A, 0)(x, h)$  still belongs to the graph of  $tL$ , and of course its projection onto  $B^d \times \{0\}$  is  $(A \cdot x, 0)$ , see Figure 5.

**Remark 2.28.** In order to fully understand the action of  $O(d, 1)$  onto co-Minkowski space, we have to describe the action of  $-\text{Id} \in O(d, 1)$  onto  $B^d \times \mathbb{R}$ . It is actually straightforward that

$$(-\text{Id}, 0)(x, h) = (x, -h). \quad (19)$$

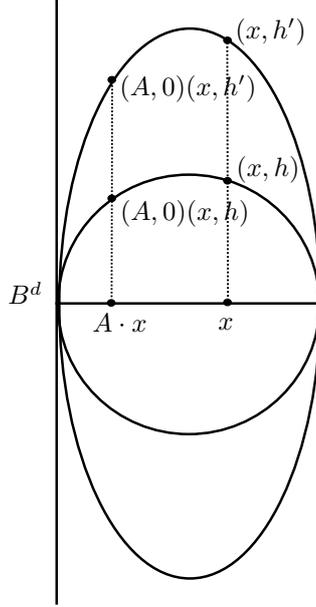


Figure 5: Action of  $(A, 0)$  on  $B^d \times \mathbb{R}$ .

We now describe the action of the isometries of co-Minkowski space on functions. Let  $S$  be a hypersurface in Minkowski space which is the graph of a map  $h : B^d \rightarrow \mathbb{R}$ . Then, for  $(A, v) \in O_+(d, 1) \times \mathbb{R}^{d,1}$ , due to (18), the hypersurface  $(A, v)S$  is the graph of the map  $(A, v)h : B^d \rightarrow \mathbb{R}$  defined as

$$(A, v)h(x) := \frac{L(x)}{L(A^{-1} \cdot x)} h(A^{-1} \cdot x) + \langle x, \bar{v} \rangle_d - v_{d+1} . \quad (20)$$

**Lemma 2.29.** *Let  $h : B^d \rightarrow \mathbb{R}$  be a  $C^2$  map and  $(A, v) \in O_+(d, 1) \times \mathbb{R}^{d,1}$ . Then*

$$\text{Hess}[(A, v)h](x)(X, Y) = \frac{L(x)}{L(A^{-1} \cdot x)} \text{Hess} h(A^{-1} \cdot x)(DA^{-1}(x)X, DA^{-1}(x)Y) .$$

*Proof.* As  $(\text{Id}, v)h$  is the sum of  $h$  with an affine function, we clearly have  $\text{Hess}[(\text{Id}, v)h](x) = \text{Hess} h(x)$ . So we need to check the result only for  $(A, 0)$ . As

$$\text{Hess}[(A, 0)h] = \text{Hess} \left( \frac{L}{L \circ A^{-1}}(h \circ A^{-1}) \right) ,$$

the result follows from the rules (11) and

$$\text{Hess}(f \circ g)(x)(X, Y) = \text{Hess} f(g(x))(d g(x)(X), d g(x)(Y)) + d f(g(x))(\text{Hess} g(x)(X, Y)) , \quad (21)$$

using the two following facts during the computations:

- $\frac{L}{L \circ A}$  is an affine map by (16), so has null Hessian;
- Differentiating two times (15) we obtain

$$A \begin{pmatrix} X \\ 0 \end{pmatrix}_{d+1} DA(x)(Y) + A \begin{pmatrix} Y \\ 0 \end{pmatrix}_{d+1} DA(x)(X) + A \begin{pmatrix} x \\ 1 \end{pmatrix}_{d+1} \text{Hess} A(x)(X, Y) = 0 ,$$

so using (16) again,

$$d \frac{L}{L \circ A} \otimes d A + d A \otimes d \frac{L}{L \circ A} + \frac{L}{L \circ A} \text{Hess} A = 0 .$$

□

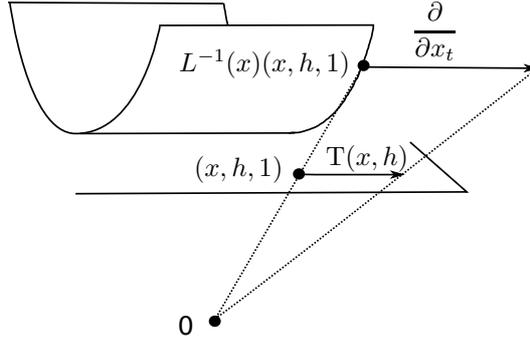


Figure 6: By Thales theorem,  $T = L \frac{\partial}{\partial x_t}$ .

**Lemma 2.30.** *Let  $h : B^d \rightarrow \mathbb{R}$  be a convex map. Then for  $(A, v) \in O_+(d, 1) \times \mathbb{R}^{d,1}$ ,  $(A, v)h$  is a convex map.*

Note that from (19),  $(-\text{Id}, 0)h$  is concave if  $h$  is convex.

*Proof.* The simplest way to see this is to argue that the dual of the epigraph of  $h$  is a future convex set in Minkowski space, see Remark 2.22. The isometry  $(A, v)$  will send this future convex set to a future convex set (because  $A \in O_+(d, 1)$ ), whose support function is exactly  $(A, v)h$ , hence convex.  $\square$

### 2.2.5 Connection in cylindrical coordinates

Clearly, the restriction of the vector field  $\frac{\partial}{\partial x_t} = (0, \dots, 0, 1)$  of  $\mathbb{R}^{d+2}$  to  $\text{coMin}^{d+1}$  is invariant under the action of the isometries of  $\text{coMin}^{d+1}$ . It is also immediate to see that  $\frac{\partial}{\partial x_t}$  is parallel:  $\nabla^{\text{coMin}^{d+1}} \frac{\partial}{\partial x_t} = 0$ . We will denote by  $T$  the image of  $\frac{\partial}{\partial x_t}$  in co-Minkowski space. An elementary computation (see Figure 6) shows that in the cylindrical coordinates  $B^d \times \mathbb{R}$ ,

$$T = L \frac{\partial}{\partial x_t} . \quad (22)$$

In particular,  $T$  is invariant under the action of  $O_+(d, 1) \times \mathbb{R}^{d,1}$ , and  $T$  is parallel:  $\nabla^{\mathbb{R}^{d,1}} T = 0$ . Observe that the trajectories of the flow generated by  $T$  are the vertical lines. In Minkowski space, the flow generated by  $T$  corresponds to parallel displacement of spacelike hyperplanes.

With the help of  $T$ , one can express the connection  $\nabla^{\mathbb{R}^{d,1}}$  in the cylindrical coordinates. Namely, at each point  $(x, h) \in B^d \times \mathbb{R}$ , we set  $T(x, h)$  as the vector basis for the  $\mathbb{R}$ -component of the tangent space. Hence a vector field  $X$  of  $B^d \times \mathbb{R}$  can be written  $X = X_h + X_T T$ , with  $X_h \in T_x B^d$  and  $X_T \in \mathbb{R}$ . If  $Y$  is another vector field of  $B^d \times \mathbb{R}$ , then

$$\nabla_Y^{\mathbb{R}^{d,1}} X = \nabla_{Y_h} X_h + Y_h(X_T)T + Y_T[T, X] . \quad (23)$$

This is easily checked using the definition of the connection  $\nabla^{\mathbb{R}^{d,1}}$  and the fact that  $T$  is parallel.

### 2.2.6 Volume form

For future reference, let us mention that a volume form  $\omega_{\text{coMin}^{d+1}}$  is also given on  $\text{coMin}^{d+1}$ . For  $v_1, \dots, v_{d+1}$  vectors of  $\mathbb{R}^{d+2}$  tangent to  $\text{coMin}^{d+1}$ , set

$$\omega_{\text{coMin}^{d+1}}(v_1, \dots, v_{d+1}) := \omega_{\mathbb{R}^{d+2}}(v_1, \dots, v_{d+1}, N)$$

(recall that  $N$  is the vector field  $N(x) = x$  on  $\text{coMin}^{d+1}$ ). This form is invariant under orientation preserving isometries and parallel for  $\nabla^{\text{coMin}^{d+1}}$ . It induces a parallel form  $\omega_{\mathbb{R}^{d,1}}$

on co-Minkowski space, invariant under orientation preserving isometries, and called the *volume form* of co-Minkowski space.

In the cylindrical coordinates,  $\omega_{*\mathbb{R}^{d,1}}$  is defined as follows. At a point of  $B^d \times \mathbb{R}$ , let  $v_1, \dots, v_d$  be an oriented free family of non-vertical tangent vectors. In particular,  $v_1, \dots, v_d$  are tangent to a hyperbolic hyperplane, so, keeping the same notation, we can consider a family  $v_1, \dots, v_d$  of oriented orthonormal vectors fields, such that  $v_1, \dots, v_d, T$  is positively oriented. Then  $\omega_{*\mathbb{R}^{d,1}}$  is the unique  $(d+1)$ -form which is equal to 1 when evaluated at such a family of vectors.

## 2.3 Extrinsic geometry of graphs

Let  $h : B^d \rightarrow \mathbb{R}$  be a  $C^2$  map. Its graph  $S$  is a hypersurface in  $B^d \times \mathbb{R}$ , hence in co-Minkowski space if one uses the cylindrical coordinates. Note that the graph is always transverse to the vertical vector field  $T$  defined by (22), so the metric induced on  $S$  by the ambient degenerate metric  $g_{*\mathbb{R}^{d,1}}$  of co-Minkowski space is always a hyperbolic metric, that does not give too much informations. But still, some informations can be obtained from the extrinsic geometry of  $S$ . To do so, we will consider the vector field  $T$  as the normal vector to  $S$ .

### 2.3.1 Second fundamental form and mean curvature

Let  $h : B^d \rightarrow \mathbb{R}$  be a  $C^2$  map and let  $S$  be its graph. Any vector field of  $S$  can be written  $X + dh(X)L^{-1}T$ , where  $X$  is a vector field of  $B^d$ .

**Fact 2.31.** *For any smooth vector field  $X$  on  $B^d$  and  $C^2$  map  $h : B^d \rightarrow \mathbb{R}$ ,*

$$\nabla_{(Y+L^{-1}dh(Y)T)}^{*\mathbb{R}^{d,1}}(X + L^{-1}dh(X)T) = \nabla_Y^{\mathbb{H}^d}X + L^{-1}dh(\nabla_Y^{\mathbb{H}^d}X)T + L^{-1}\text{Hess } h(X, Y)T . \quad (24)$$

*Proof.* First let  $k \in \{1, \dots, d\}$ . As  $X$  does not depend on the  $\frac{\partial}{\partial x_i}$  direction, and as  $T^k = 0$ ,  $[T, X]^k = T^i \frac{\partial X^k}{\partial x_i} - X^i \frac{\partial T^k}{\partial x_i} = 0$ , and  $[T, X]^t = -X^i \frac{\partial T^t}{\partial x_i} = -X(L)$ . Also, as  $L^{-1}dh(X)$  does not depend on the vertical coordinate,  $[T, L^{-1}dh(X)T] = 0$ . At the end of the day, if we are at a point  $x \in B^d$ ,

$$[T, X + L^{-1}dh(X)T] = -X(L) \frac{\partial}{\partial x_i} = -X(L)L^{-1}T = L^{-2}\langle x, X \rangle T .$$

So from (23),

$$\nabla_{(Y+L^{-1}dh(Y)T)}^{*\mathbb{R}^{d,1}}(X + L^{-1}dh(X)T) = \nabla_Y X + (Y(L^{-1}dh(X)) + L^{-3}dh(Y)\langle x, X \rangle_d)T .$$

We have  $Y(L^{-1}dh(X)) = L^{-1}(Y(X(h)) + Y(L^{-1})dh(X))$ , and from (10),  $L^{-1}(Y(X(h))) = L^{-1}\text{Hess}^{\mathbb{H}^d}h(X, Y) + L^{-1}dh(\nabla_Y X)$ . Also, if we are at the point  $x$ ,  $Y(L^{-1}) = \langle x, Y \rangle_d L^{-3}$ :

$$\nabla_{(Y+L^{-1}dh(Y)T)}^{*\mathbb{R}^{d,1}}(X + L^{-1}dh(X)T) = \nabla_Y X + L^{-1}dh(\nabla_Y X)T + L^{-1}\mathcal{X}T$$

with  $\mathcal{X} = \text{Hess}^{\mathbb{H}^d}h(X, Y) + L^{-2}\langle x, Y \rangle_d dh(X) + L^{-2}\langle x, X \rangle_d dh(Y)$ , and by (8),  $\mathcal{X} = \text{Hess } h(X, Y)$ .  $\square$

Given two vector fields tangent to  $S$ , the graph of  $h$ , then their co-Minkowski connection decomposes as a part tangent to  $S$ , and a part colinear to  $T$ , where  $T$  may be think as a unit normal vector field to  $S$ . Mimicking the classical theory of surfaces, we define the *second fundamental form*  $\mathbb{I}_h$  of  $S$  as the colinearity factor. More precisely, Equation (24) says that for  $x \in B^d$  and  $X, Y \in T_x B^d$ ,

$$\mathbb{I}_h(x)(X, Y) = L^{-1}(x)\text{Hess } h(x)(X, Y) . \quad (25)$$

**Remark 2.32.** From Lemma 2.13, the second fundamental form is a symmetric Codazzi tensor on  $\mathbb{H}^d$ , and any symmetric Codazzi tensor on the hyperbolic space is the second fundamental form of a unique hypersurface in co-Minkowski space. This is a kind of “fundamental theorem for hypersurfaces” in co-Minkowski space, with the condition about the

first fundamental form reduced to the hypothesis that the metric is hyperbolic. Note that here there is no Gauss condition, i.e. for  $d = 2$  there is no relation between the curvature of the induced metric and the determinant of the second fundamental form.

The *shape operator*  $\mathbf{shape}(h)$  of  $S$  is the symmetric linear mapping associated to the second fundamental form by the hyperbolic metric:  $\mathbb{I}_h(X, Y) = g_{\mathbb{H}^d}(\mathbf{shape}(h)(X), Y)$ . From (9), if  $\text{grad}^{\mathbb{H}^d}$  is the gradient for  $g_{\mathbb{H}^d}$ , we have

$$\mathbf{shape}(h)(X) = \nabla_X^{\mathbb{H}^d} \text{grad}^{\mathbb{H}^d}(L^{-1}h) - (L^{-1}h)X .$$

The *mean curvature*  $\mathbf{Mean}(h)$  of the graph of  $h$  is the trace for the hyperbolic metric of the shape operator times  $1/d$ . From the definition or Fact 2.8, it can be written in different ways: with the help of the the Euclidean Laplacian  $\Delta$

$$\mathbf{Mean}(h)(x) = \frac{1}{d} \text{Tr}_{g_{\mathbb{H}^d}}(L^{-1}\text{Hess } h)(x) = \frac{1}{d} L(x)(\Delta h(x) - \text{Hess } h(x)(x, x)) , \quad (26)$$

or with the help of the hyperbolic Laplacian  $\Delta^{\mathbb{H}^d}$

$$\mathbf{Mean}(h)(x) = \frac{1}{d} \Delta^{\mathbb{H}^d}(L^{-1}h)(x) - (L^{-1}h)(x) . \quad (27)$$

**Remark 2.33.** Let us suppose that  $h : B^d \rightarrow \mathbb{R}$  is  $C^2$  and convex. Using a basis of eigenvectors, it follows from (26) that  $\mathbf{Mean}(h)$  is non-negative, and that if  $\mathbf{Mean}(h) = 0$  then  $h$  is affine.

**Proposition 2.34** ([Li95]). *If the graph of a  $C^2$  convex function  $h : B^d \rightarrow \mathbb{R}$  has its mean curvature bounded from above, then  $h$  has a continuous extension to  $\bar{B}^d$ .*

*Proof.* Suppose that there is  $C$  such that for any  $x \in B^d$ ,  $\mathbf{Mean}(h)(x) < C$ . Let  $\theta \in \partial B^d$ , and let  $h_\theta$  be the restriction of  $h$  to the segment parametrized by  $r \in [0, 1[$  from the origin to  $\theta$ . Let us also denote  $l(r) = \sqrt{1 - r^2}$ . By (26),

$$h_\theta''(r) < Cl(r)^{-3} .$$

For  $1/2 < r < 1$ , we write

$$h_\theta'(r) \leq h_\theta'(1/2) + C \int_{1/2}^r l^{-3}$$

and as, for  $1/2 < t < 1$ ,  $(1 - t^2)^{-1} < (1 - t)^{-1}$ , we have

$$h_\theta'(r) < h_\theta'(1/2) + 2C(1 - r)^{-1/2} . \quad (28)$$

Also, as  $h$  is convex,  $h_\theta$  is convex, hence for  $1/2 < r < 1$ ,

$$h_\theta'(1/2) \leq h_\theta'(r) . \quad (29)$$

Let us define

$$g(\theta) = \int_{1/2}^1 h_\theta' - h_\theta(1/2) .$$

As  $\int_{1/2}^1 (1 - r)^{-1/2} dr$  is finite, by (28) and (29),  $g(\theta)$  is well defined. Also, together with (28), (29) and the Dominated convergence theorem,  $g$  is continuous.  $\square$

### 2.3.2 Mean surfaces

**Definition 2.35.** *A hypersurface  $S$  of co-Minkowski space is called mean if it is the graph of a  $C^2$  function  $h : B^d \rightarrow \mathbb{R}$  with  $\mathbf{Mean}(h) = 0$ .*

*Abusing terminology, the function  $h$  itself may be also called mean.*

Note that when  $d = 2$ , the mean surface is not critical for the area functional, as all the graphs of functions  $B^2 \rightarrow \mathbb{R}$  in co-Minkowski space have the same area form (because they are all isometric to the hyperbolic plane).

Due to (26),  $h$  is mean if and only if for any  $x \in B^d$ ,  $\Delta h(x) - \text{Hess } h(x)(x, x) = 0$ . This is an elliptic equation with only second-order terms, that allows to apply strong results of PDE theory. For this, we have to consider boundaries conditions.

**Definition 2.36.** Let  $b : \partial B^d \rightarrow \mathbb{R}$  be a continuous map. A continuous function  $h : B^d \rightarrow \mathbb{R}$  is called a  $b$ -map if it extends continuously as  $b$  on  $\partial B^d$ .

**Proposition 2.37.** For any continuous function  $b : \partial B^d \rightarrow \mathbb{R}$ , there is a unique  $C^\infty$  smooth mean  $b$ -map, denoted by  $h_b^{\text{mean}}$ .

*Proof.* The uniqueness is classical from the ellipticity of  $L^{-1}\mathbf{Mean}$  [GT01, Theorem 3.3]. Existence follows from the fact that the elliptic equation  $\Delta f(x) - \text{Hess } f(x)(x, x) = 0$  has only second-order terms and that the domain is a ball, see [GT01, Corollary 6.24']. Dividing the equation by  $L^2$ , we obtain a strictly elliptic equation, and regularity theorems apply, e.g. [GT01, Corollary 8.11].  $\square$

**Lemma 2.38.** If  $b_n : \partial B^d \rightarrow \mathbb{R}$  are continuous functions uniformly converging to  $b : \partial B^d \rightarrow \mathbb{R}$ , then  $h_{b_n}^{\text{mean}}$  is converging to  $h_b^{\text{mean}}$ .

*Proof.* Let  $b_n$  such that the supremum of  $|b_n - b|$  is arbitrarily small. Then  $\mathbf{Mean}(h_{b_n}^{\text{mean}} - h_b^{\text{mean}}) = 0$ , with boundary data  $b_n - b$ . By the maximum principle [GT01, Theorem 3.1],  $h_{b_n}^{\text{mean}} - h_b^{\text{mean}}$  is arbitrarily small. The same conclusion holds for  $h_b^{\text{mean}} - h_{b_n}^{\text{mean}}$ .  $\square$

**Remark 2.39.** For a continuous map  $b : \partial B \rightarrow \mathbb{R}$ , it is possible to associate to  $h_b^{\text{mean}}$  a (non-regular and non convex) dual hypersurface in Minkowski space, see Remark 2.24. For  $d = 2$ , at points of regularity, this surface has zero mean curvature. We refer to [FV16] for more details.

### 2.3.3 Convex hull

Let  $b : \partial B^d \rightarrow \mathbb{R}$  be a continuous map. Let

$$\mathcal{A}_b = \{a : \mathbb{R}^d \rightarrow \mathbb{R} \text{ is an affine function and } a|_{\partial B^d} \leq b\}$$

and for  $x \in B^d$ , let us define

$$h_b^-(x) := \sup\{a(x) | a \in \mathcal{A}_b\}, \quad (30)$$

and

$$h_b^+(x) := -h_{-b}^-(x). \quad (31)$$

**Proposition 2.40.** For any  $x \in B^d$ ,  $h_b^-(x)$  defines a convex  $b$ -map  $h_b^- : B^d \rightarrow \mathbb{R}$ . Moreover, if  $h : B^d \rightarrow \mathbb{R}$  is a convex  $b$ -map, then  $h_b^- \geq h$ .

For any  $x \in B^d$ ,  $h_b^+(x)$  defines a concave  $b$ -map  $h_b^+ : B^d \rightarrow \mathbb{R}$ . Moreover, if  $h : B^d \rightarrow \mathbb{R}$  is a concave  $b$ -map, then  $h_b^+ \leq h$ .

In general, we have  $h_b^+ \geq h_b^-$ . If  $h_b^+ = h_b^-$ , then  $b$  is the restriction to  $\partial B^d$  of an affine map of  $\mathbb{R}^d$ .

*Proof.* The properties of  $h_b^-$  are proved in the proof of Theorem 1.5.2 in [Gut01]. The properties of  $h_b^+$  then follows immediately from (31). The last property is then obvious, as affine maps are the only ones being in the same time convex and concave.  $\square$

Let  $\Lambda(b)$  be the graph of  $b : \partial B^d \rightarrow \mathbb{R}$  in  $\partial B^d \times \mathbb{R}$ , and let  $\text{CH}(b)$  be the affine convex hull of  $\Lambda(b)$  in  $\mathbb{R}^{d+1}$ , that is, the smallest convex set of  $\mathbb{R}^{d+1}$  containing  $\Lambda(b)$ . Note that as  $\bar{B}^d \times \mathbb{R}$  is a convex set containing  $\Lambda(b)$ , then  $\text{CH}(b) \subset \bar{B}^d \times \mathbb{R}$ .

**Lemma 2.41.** The boundary of  $\text{CH}(b)$  is the union of the graphs of  $h_b^+$  and  $h_b^-$ .

*Proof.* This follows from the definitions of  $h_b^+$  and  $h_b^-$ , because  $\text{CH}(b)$  is the intersection of all the half-spaces containing  $\Lambda(b)$ .  $\square$

The set  $\text{CH}(b)$  satisfies the local geodesic property: for any  $x \in \text{CH}(b) \setminus \Lambda(b)$ ,  $x$  lies in an open segment contained in  $\text{CH}(b) \setminus \Lambda(b)$  [Smi15, Theorem 4.19].

**Lemma 2.42.** *The mean surface given by the boundary condition  $b : \partial B^d \rightarrow \mathbb{R}$  is contained in the convex hull  $\text{CH}(b)$ :*

$$h_b^- \leq h_b^{\text{mean}} \leq h_b^+ .$$

*Proof.* Let  $a \in \mathcal{A}_b$ . By the maximum principle [GT01, Theorem 3.1],  $a - h_b^{\text{mean}}$  attains its maximal value on  $\partial B$ . But on  $\partial B^d$ ,  $a \geq b$ , so on  $B^d$ ,  $a - h_b^{\text{mean}} \leq a|_{\partial B^d} - b = b - b = 0$ , i.e.  $a \leq h_b^{\text{mean}}$ . Then by definition of  $h_b^-$ ,  $h_b^- \leq h_b^{\text{mean}}$ . Similarly, one proves that  $h_{-b}^- \leq h_{-b}^{\text{mean}} = -h_b^{\text{mean}}$  i.e.  $h_b^+ = -h_{-b}^- \geq h_b^{\text{mean}}$ .  $\square$

**Lemma 2.43.** *If  $(b_n)_{n \in \mathbb{N}}$  is a sequence of continuous functions from  $\partial B^d$  into  $\mathbb{R}$  converging uniformly to  $b : \partial B^d \rightarrow \mathbb{R}$ , then  $(h_{b_n}^-)_{n \in \mathbb{N}}$  (resp.  $(h_{b_n}^+)_{n \in \mathbb{N}}$ ) is converging to  $h_b^-$  (resp.  $h_b^+$ ).*

*Proof.* Let  $\epsilon > 0$  and  $x \in B^d$ . Then there exists an affine function  $a$  such that  $h_b^-(x) \geq a(x)$ ,  $a(x) + \epsilon \geq h_b^-(x)$  and  $a|_{\partial B^d} \leq b$ . In particular, for  $n$  large enough,  $a|_{\partial B^d} - \epsilon \leq b_n$ . As  $a|_{\partial B^d} - \epsilon$  is an affine function, then  $h_{b_n}^-(x) \geq a(x) - \epsilon$ . As  $a$  was chosen such that  $a(x) + \epsilon \geq h_b^-(x)$ , then  $h_{b_n}^-(x) + 2\epsilon \geq h_b^-(x)$ . A similar conclusion holds, exchanging the roles of  $b$  and  $b_n$ .  $\square$

**Remark 2.44.** The dual in Minkowski space of the epigraph of a convex  $b$ -map is a convex set. Its *domain of dependence*, or *Cauchy domain*, denoted by  $\Omega_b^-$ , is the interior of the intersection of the future side of all the lightlike hyperplanes containing it. This intersection is nothing but the dual of the the epigraph of  $h_b^-$ . The domain of dependence  $\Omega_b^-$  is future complete. Considering  $h_b^+$  instead of  $h_b^-$ , and concave figures instead of convex ones, we obtain the domain of dependence  $\Omega_b^+$ . See Figure 8 and [Bar05, Bon05] for more details.

**Remark 2.45.** The function  $h_b^{\text{mean}}$  is the solution of the Dirichlet problem for an elliptic linear equation. The convex function  $h_b^-$  is the solution of the Dirichlet problem for the Monge–Ampère equation, see [Gut01].

### 2.3.4 The mean curvature measure

For a  $C^2$  function  $h : B^d \rightarrow \mathbb{R}$ , we have defined in Section 2.3.1 the mean curvature function, which is non-negative if  $h$  is convex by Remark 2.33. For a convex  $C^2$  function  $h : B^d \rightarrow \mathbb{R}$ , let us define the *mean curvature measure*

$$\mathbf{MM}(h) = d\mathbf{Mean}(h)\omega_{\mathbb{H}^d} ,$$

where  $\omega_{\mathbb{H}^d}$  is the volume form given by the hyperbolic metric on  $B^d$ . By (6) and (26), for any  $\varphi \in C_0^0(B^d)$  (here the subscript 0 means “with compact support”),

$$\mathbf{MM}(h)(\varphi) = \int_{B^d} (\Delta h(x) - \text{Hess } h(x)(x, x)) L^{-d}(x) \varphi(x) \, dx .$$

If moreover  $\varphi \in C_0^\infty(B^d)$ , by integration by part:

$$\mathbf{MM}(h)(\varphi) = \int_{B^d} (\Delta \varphi(x) - \text{Hess } \varphi(x)(x, x)) h(x) L^{-d}(x) \, dx . \quad (32)$$

For any convex function  $h : B^d \rightarrow \mathbb{R}$ , let us define  $\mathbf{MM}(h)$  as the linear form on  $C_0^\infty(B^d)$  defined by (32). On any compact ball  $K$  contained in  $B^d$ , by standard convolution, one can find a sequence  $(h_i)_{i \in \mathbb{N}}$  of  $C^\infty$  convex functions uniformly approximating  $h$ . For any  $C^\infty$  function  $\varphi$  whose support is included in  $K$ , we clearly have  $\mathbf{MM}(h_j)(\varphi) \rightarrow \mathbf{MM}(h)(\varphi)$ . As  $\mathbf{MM}(h_j)$  is a measure, it is also a distribution, and the preceding limit says that  $\mathbf{MM}(h)$  is also a distribution on  $K$  [Hö3, Theorem 2.1.8]. Actually, as the  $\mathbf{MM}(h_j)$  are measures, then  $\mathbf{MM}(h)$  is a measure on  $K$  [Hö3, Theorem 2.1.9, Theorem 2.1.7]. Changing  $K$  and using the localization property of distribution [Hö3, Theorem 2.2.4], it follows that  $\mathbf{MM}(h)$  is a measure on  $B^d$ . More precisely,  $\mathbf{MM}(h)$  is a Radon measure on  $B^d$ .

The following result is given by [Hö3, Theorem 2.1.9, Theorem 2.1.7].

**Lemma 2.46.** *Let  $(h_n)_{n \in \mathbb{N}}$  be a sequence of convex functions from  $B^d$  into  $\mathbb{R}$  converging to a convex function  $h : B^d \rightarrow \mathbb{R}$ . Then the sequence of measures  $(\mathbf{MM}(h_n))_{n \in \mathbb{N}}$  weakly converges to  $\mathbf{MM}(h)$ .*

Recall the action of isometries on functions defined by (20). Recall also from Lemma 2.30 that if  $h$  is convex, then  $(A, v)h$  is convex for  $(A, v) \in O_+(d, 1) \times \mathbb{R}^{d,1}$ .

**Lemma 2.47.** *Let  $\varphi \in C_0^0(B^d)$  and  $(A, v) \in O_+(d, 1) \times \mathbb{R}^{d,1}$ . Then:*

$$\mathbf{MM}((A, v)h)(\varphi) = \mathbf{MM}(h)(\varphi \circ A) .$$

*Proof.* We will prove the result for a  $C^2$  function  $h$ , the general result follows by approximation. In the  $C^2$  case, the result follows because by definition

$$\mathbf{MM}(h)(\varphi \circ A) = \int_{B^d} (\varphi \circ A)(x) (\mathrm{Tr}_{g_{\mathbb{H}^d}} L^{-1} \mathrm{Hess} h)(x) d\omega_{\mathbb{H}^d}(x)$$

so by a change of variable, as  $A$  is a hyperbolic isometry,

$$\mathbf{MM}(h)(\varphi \circ A) = \int_{B^d} \varphi(x) (\mathrm{Tr}_{g_{\mathbb{H}^d}} L^{-1} \mathrm{Hess} h)(A^{-1} \cdot x) d\omega_{\mathbb{H}^d}(x) ,$$

and by Lemma 2.29,

$$(\mathrm{Tr}_{g_{\mathbb{H}^d}} L^{-1} \mathrm{Hess} h)(A^{-1} \cdot x) = (\mathrm{Tr}_{g_{\mathbb{H}^d}} L^{-1} \mathrm{Hess} [(A, v)h])(x) .$$

□

### 2.3.5 The fundamental example of a wedge

Let us consider an elementary example to give a geometric insight on the mean curvature measure introduced in the previous section. This example will make clear that, for well-chosen convex functions, this measure is a kind of "pleating measure", similar to the notion developed by Thurston for isometric pleated embeddings of hyperbolic surfaces in the 3-dimensional hyperbolic space, see sections 3.3.4 and 3.4.

Let  $l$  be the intersection of  $B^d$  with an affine hyperplane of  $\mathbb{R}^d$ , which separates  $B^d$  into two connected components  $l^-$  and  $l^+$ , where  $l^-$  is the component containing the origin 0 of the coordinates of  $\mathbb{R}^d$ . Let  $p_l$  be the (Euclidean) orthogonal projection of 0 onto  $l$ , and let  $n_l = p_l / \|p_l\|$ . If  $l$  is a vector hyperplane, then  $l^-$  is chosen arbitrarily, and  $n_l$  is the (Euclidean) unit normal vector pointing to  $l^+$ .

**Definition 2.48.** *The canonical map  $h_l : B^d \rightarrow \mathbb{R}$  associated to  $l$  is defined as  $h_l(x) = \frac{1}{L(p_l)} \langle x - p_l, n_l \rangle$ .*

Observe that  $h_l$  is an affine map vanishing on  $l$ . Let  $\mathbf{1}_A$  be the indicator function of a set  $A$ .

**Definition 2.49.** *A wedge on a hyperplane  $l$  is a continuous map  $h : B^d \rightarrow \mathbb{R}$  of the form  $h = h_- + (h_+ - h_-)\mathbf{1}_{l^+}$  where  $h_-, h_+$  are two affine maps.*

*The angle of a wedge (in the co-Minkowski sense) is the unique real number  $\alpha$  such that, with the notations above,*

$$h_+ - h_- = \alpha h_l . \tag{33}$$

The wedge is therefore a piecewise affine map, admitting  $l$  as a locus of non-differentiability (if the angle is nonzero).

**Fact 2.50.** *A wedge is convex and different from an affine map if and only its angle is positive.*

*Proof.* By definition,  $h_l$  is positive on  $l^+ \setminus l$ . And  $h$  is strictly convex if and only if on  $l^+ \setminus l$ ,  $h_+ = h_- + \alpha h_l > h_-$ , that is true if and only if  $\alpha > 0$ . □

**Remark 2.51.** The hyperplane  $l$  in  $B^d$  defines a timelike vector hyperplane in Minkowski space, namely, if  $B^d$  is identified with the Klein ball model of the hyperbolic space in  $\mathbb{R}^{d,1}$ , the vector hyperplane passing through  $l \times \{1\}$ . Let  $v_l$  be its unit spacelike normal vector pointing to the side containing  $l^+$ . Then it is easy to see that

$$v_l = \frac{1}{p_l} \begin{pmatrix} n_l \\ \|p_l\| \end{pmatrix}, \quad (34)$$

and so the canonical map  $h_l$  is the restriction to  $B^d \times \{1\}$  of the linear map  $(x, x_{d+1}) \mapsto \langle \begin{pmatrix} x \\ x_{d+1} \end{pmatrix}, v_l \rangle_{d,1}$ . If  $l$  is a vector hyperplane, then  $v_l = \begin{pmatrix} n_l \\ 0 \end{pmatrix}$ . Moreover, if  $P_+$  and  $P_l$  are the duals of the graphs of  $h_+$  and  $h_-$ , then  $P_+ - P_-$  is colinear to  $v$ , that expresses the definition (33) (compare also with Fact 2.21). The absolute value of  $\alpha$  is the Minkowski length of the spacelike segment  $P_+ - P_-$ . See Figure 7.

**Fact 2.52.** Let  $A \in O_+(d, 1)$ . Then  $h_{A \cdot l} = \frac{L}{L \circ A^{-1}} h_l \circ A^{-1}$ .

*Proof.* With the notations of Remark 2.51, we clearly have  $v_{A \cdot l} = A(v_l)$ , hence  $h_{A \cdot l}(x) = \langle \begin{pmatrix} x \\ 1 \end{pmatrix}, A(v_l) \rangle_{d,1} = \langle A^{-1} \begin{pmatrix} x \\ 1 \end{pmatrix}, v_l \rangle_{d,1}$ , and by (17),  $A^{-1} \begin{pmatrix} x \\ 1 \end{pmatrix} = \frac{L(x)}{L(A^{-1} \cdot x)} \begin{pmatrix} A^{-1} \cdot x \\ 1 \end{pmatrix}$ .  $\square$

**Fact 2.53.** The image of the graph of a wedge by an orientation preserving co-Minkowski isometry is the graph of a wedge of same angle.

*Proof.* The result is obvious from Remark 2.51, as a co-Minkowski isometry acts as a Minkowski isometry on the dual objects, and hence sends a spacelike segment to a spacelike segment of same length.  $\square$

The choice of the normal  $n_l$  gives an orientation on the vector hyperplane  $l$ , which is also isometric to  $\mathbb{H}^{d-1}$ . We denote by  $\omega_l^{\mathbb{H}}$  its volume form for the hyperbolic metric.

**Lemma 2.54.** Let  $h$  be a convex wedge of angle  $\alpha$  on a hyperplane  $l$ . Then the following identity holds:

$$\mathbf{MM}(h) = \alpha \omega_l^{\mathbb{H}}.$$

The simplest illustration of the lemma is for  $d = 1$ ,  $l = \{0\}$  and  $h(x) = |x| = -x + 2x\mathbf{1}_{\mathbb{R}^+}$ . Then the angle is equal to 2, and  $h''$  in the sense of distributions is equal to  $2\delta(0)$ .

*Proof.* From (33),  $h = h_- + \alpha h_l \mathbf{1}_{l^+}$ , so as  $h_-$  is affine, in the sense of distributions,  $\partial_{ij} h = \alpha \partial_{ij} (h_l \mathbf{1}_{l^+})$ . By successive integrations by part, for  $\phi \in C_0^\infty$ , using that  $h_l = 0$  on  $l$  and that  $h_l$  is affine, we obtain, in the sense of distributions,  $\partial_{ij} h = \alpha \partial_i h_l (n_l)_j dS$ , where  $dS$  is the (Euclidean) area form on  $l$  ( $n_l$  is an inward normal vector for  $l^+$ ).

Hence by (32) and (33), the measure  $\mathbf{MM}(h)$  is given by, for  $x \in l$ ,

$$\alpha \langle n_l, \text{grad } h_l \rangle + \langle n_l, x \rangle \langle \text{grad } h_l, x \rangle L^{-d}(x) dS(x). \quad (35)$$

Let us first consider that  $l$  is the intersection of  $B^d$  with a vector hyperplane. Then,  $\langle n_l, x \rangle = 0$ ,  $\langle n_l, \text{grad } h_l \rangle = 1$ , and from (6),  $L^{-d}(x) dS = d\omega_l^{\mathbb{H}}$ . At the end of the day, (35) becomes  $\alpha d\omega_l^{\mathbb{H}}$ , that is the wanted result when  $l$  is defined by a vector hyperplane. The general case follows by performing an orientation-preserving isometry sending  $l$  to a vector hyperplane, and using Lemma 2.47 and Fact 2.53.  $\square$

**Remark 2.55.** Given a hyperplane  $l$  of  $B^d$  weighted by a positive number  $\alpha$ , it is almost clear how to construct a convex wedge in co-Minkowski space with angle  $\alpha$ . This construction can be easily extended to non intersecting weighted hyperplanes (see Section 3.3.4), or to a ‘‘polyhedral case’’, i.e. weighted hyperplanes are allowed to meet to form a convex cellulation of  $\mathbb{H}^d$ , together with a natural compatibility conditions at the weights, see [FV16, 4.4] and [FS17] for the  $d = 2$  case. This is a polyhedral version of the *Christoffel problem*, whose aim is to find a convex hypersurface in Minkowski space prescribing the dual Mean curvature measure — called the *area measure of order one* in this setting. The Christoffel problem in Minkowski space is the subject of [FV16].

The polyhedral construction is also a version of the classical *Maxwell-Cremona correspondence* or *Maxwell lift*, see [Izm18].

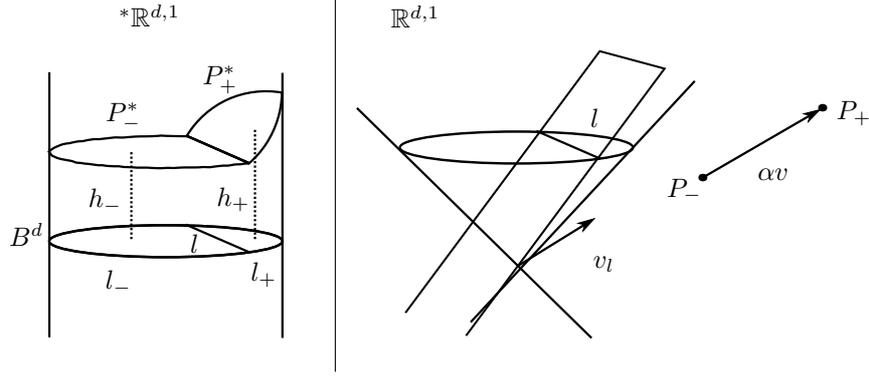


Figure 7: The spacelike segment in Minkowski space dual to a convex wedge in co-Minkowski space.

### 3 Action of cocompact hyperbolic isometry groups

#### 3.1 Translation parts as cocycles

Let  $\Gamma$  be a subgroup of  $O_+(d, 1)$  such that  $\mathcal{H}^d/\Gamma$  is a compact oriented hyperbolic manifold. A *cocycle*  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$  is a map  $\tau : \Gamma \rightarrow \mathbb{R}^{d,1}$  satisfying, for  $A, B \in \Gamma$ ,

$$\tau(AB) = \tau(A) + A(\tau(B)) .$$

Let us denote

$$\Gamma_\tau = \{(A, \tau(A)) | A \in \Gamma\} .$$

From (2),  $\Gamma_\tau$  is a subgroup of the isometry group of Minkowski space. In turn, it defines a group of isometries of co-Minkowski space, that we will also denote by  $\Gamma_\tau$ .

In the cylindrical coordinates  $B^d \times \mathbb{R}$  of co-Minkowski space,  $\Gamma$  acts freely and properly discontinuously on  $B^d \times \{0\}$ . As co-Minkowski space is the product manifold  $B^d \times \mathbb{R}$ , due to (18), the following result is trivial, but worth to notice.

**Lemma 3.1.** *The action of  $\Gamma_\tau$  on  ${}^*\mathbb{R}^{d,1}$  is free and properly discontinuous.*

A *coboundary* is a particular cocycle of the form

$$\tau(A) = Av - v$$

for a given  $v \in \mathbb{R}^{d,1}$ . The group  $H^1(\Gamma, \mathbb{R}^{d,1})$  is the quotient of the space of cocycles by the space of coboundaries: two cocycles are in relation if and only if they differ by a coboundary.

In the following, we make the implicit assumption that we are looking at  $\Gamma$  such that  $H^1(\Gamma, \mathbb{R}^{d,1})$  is not reduced to zero.

Let us give a criterion of non-triviality. Let us suppose that the compact hyperbolic manifold  $\mathcal{H}^d/\Gamma$  contains  $n$  disjoint embedded totally geodesic hypersurfaces  $H_1, \dots, H_n$ . Also, let us set some positive weights  $\omega_i$  to each  $H_i$ . This is actually a *simplicial measured geodesic lamination*  $\lambda$  on  $\mathcal{H}^d/\Gamma$ .

A lift to  $B^d$  of a  $H_i$  is a hyperplane  $l$ . Recall from (34) that a vector  $v_l$  of  $\mathbb{R}^{d,1}$  is assigned to any such  $l$ . Let us denote by  $\tilde{L}$  the set of the lifts the  $H_i$ . Let us fix an arbitrary base point  $\tilde{x} \in B^d \setminus \tilde{L}$ . Then define, for  $A \in \Gamma$ , and for any path  $c : [0, 1] \rightarrow B^d$ , transverse to  $\tilde{L}$  and joining  $\tilde{x}$  to  $A \cdot \tilde{x}$ :

$$\tau_\lambda(A) = \sum_{j \in c([0,1]) \cap \tilde{L}} \omega_j v_j . \quad (36)$$

Clearly, the definition of  $\tau_\lambda$  is independent from the choice of the path  $c$  among paths transverse to  $\tilde{L}$  joining the same endpoints.

**Fact 3.2.** *With the notations above,  $\tau_\lambda \in Z^1(\Gamma, \mathbb{R}^{d,1})$ .*

*Proof.* Let  $A, B \in \Gamma$ . Let  $c_A, c_B : [0, 1] \rightarrow B^d$  be paths transverse to  $\tilde{L}$ , and joining  $\tilde{x}$  to  $A \cdot \tilde{x}$  and  $B \cdot \tilde{x}$  respectively. Let  $c_{AB}$  be the concatenation of  $c_A$  with  $A \cdot c_B$ . This is a path joining  $\tilde{x}$  to  $(AB) \cdot \tilde{x}$  and transverse to  $L$ , so

$$\tau_\lambda(AB) = \sum_{j \in (c_{AB}([0,1]) \cap \tilde{L})} \omega_j v_j = \sum_{j \in (c_A([0,1]) \cap \tilde{L})} \omega_j v_j + \sum_{j \in (A \cdot c_B([0,1]) \cap \tilde{L})} \omega_j v_j .$$

By definition of  $v_l$ , we clearly have  $v_{A \cdot l} = A(v_l)$ , and  $A$  acts linearly on  $\mathbb{R}^{d,1}$ , so

$$\tau_\lambda(AB) = \sum_{j \in (c_A([0,1]) \cap \tilde{L})} \omega_j v_j + A \left( \sum_{j \in (c_B([0,1]) \cap \tilde{L})} \omega_j v_j \right) = \tau(A) + A(\tau(B)) .$$

□

**Fact 3.3.** *Let  $\tau'_\lambda$  be the cocycle defined by (36), but choosing another basepoint  $\tilde{x}'$ . Then  $\tau'_\lambda - \tau_\lambda$  is a coboundary.*

*Proof.* For any  $A \in \Gamma$ , let  $c : [0, 1] \rightarrow B^d$  be a path transverse to  $\tilde{L}$  joining  $\tilde{x}$  to  $A \cdot \tilde{x}$ , and let  $c' : [0, 1] \rightarrow B^d$  be a path transverse to  $\tilde{L}$  joining  $\tilde{x}'$  to  $A \cdot \tilde{x}'$ . Let  $\bar{c} : [0, 1] \rightarrow B^d$  be any path transverse to  $\tilde{L}$  joining  $\tilde{x}$  to  $\tilde{x}'$ . Then the concatenation  $c^*$  of  $\bar{c}$  with  $c'$  and  $-A \cdot \bar{c}$  is a transverse path joining  $\tilde{x}$  to  $A \cdot \tilde{x}$ , so

$$\begin{aligned} \tau_\lambda(A) &= \sum_{j \in (c^*([0,1]) \cap \tilde{L})} \omega_j v_j = \sum_{j \in (\bar{c}([0,1]) \cap \tilde{L})} \omega_j v_j + \sum_{j \in (c'([0,1]) \cap \tilde{L})} \omega_j v_j - \sum_{j \in (A \cdot \bar{c}([0,1]) \cap \tilde{L})} \omega_j v_j \\ &= \sum_{j \in (\bar{c}([0,1]) \cap \tilde{L})} \omega_j v_j + \tau'_\lambda(A) - A \left( \sum_{j \in (\bar{c}([0,1]) \cap \tilde{L})} \omega_j v_j \right) \end{aligned}$$

so if  $v$  is the vector  $-\sum_{j \in (\bar{c}([0,1]) \cap \tilde{L})} \omega_j v_j$  we have  $\tau_\lambda(A) - \tau'_\lambda(A) = Av - v$ . □

So for each choice of positive weights, we have constructed an element of  $H^1(\Gamma, \mathbb{R}^{d,1})$ . Clearly, a linearly independent change in the weights will produce a different element in  $H^1(\Gamma, \mathbb{R}^{d,1})$ , hence we have a simple geometric proof of the following classical result (see the Introduction).

**Theorem 3.4.** *If  $\mathbb{H}^d/\Gamma$  contains  $n$  disjointly embedded totally geodesic hypersurfaces, then the dimension of  $H^1(\Gamma, \mathbb{R}^{d,1})$  is  $\geq n$ .*

### 3.2 Equivariant maps

Let  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ . We will give more details on the action of  $\Gamma_\tau$  by looking at particular functions. The analysis is simplified using the cylindrical coordinates of co-Minkowski space. We say that a continuous map  $h : B^d \rightarrow \mathbb{R}$  is  $\Gamma$ -invariant if its graph is invariant for the action of  $\Gamma$ , i.e. for all  $A \in \Gamma$ ,  $(A, 0)h = h$  (recall (20)):

$$\forall x \in B^d, (L^{-1}h)(A \cdot x) = (L^{-1}h)(x) ,$$

in other terms,  $h$  is  $\Gamma$ -invariant if and only if  $L^{-1}h$  is invariant for the action of  $\Gamma$ . In particular, if  $h$  is  $\Gamma$ -invariant, as the action of  $\Gamma$  is cocompact on  $B^d$ ,  $L^{-1}h$  is bounded. Note that the function  $L$  is obviously  $\Gamma$ -invariant (see Remark 2.27 for a geometric viewpoint).

**Fact 3.5.** *Let  $h$  be a  $\Gamma$ -invariant function. Then  $h$  extends continuously as the constant zero function on  $\partial B^d$ .*

*Proof.* There exists two constants  $c_1, c_2$  such that  $c_1 \leq L^{-1}h \leq c_2$ , so  $c_1 L \leq h \leq c_2 L$ , and the result follows. □

**Definition 3.6.** A continuous map  $h : B^d \rightarrow \mathbb{R}$  is  $\tau$ -equivariant if its graph is invariant for the action of  $\Gamma_\tau$ , i.e. for all  $A \in \Gamma$ ,  $(A, \tau(A))h = h$ , using the notation introduced in (20).

The vector space structure of  $Z^1(\Gamma, \mathbb{R}^{d,1})$  fits well with the vector space structure of maps, as the following lemma shows. Its proof is trivial from Definition 3.6.

**Fact 3.7.** Let  $\tau_1, \tau_2 \in Z^1(\Gamma, \mathbb{R}^{d,1})$  and let  $h_1$  and  $h_2$  be  $\tau_1$  and  $\tau_2$ -equivariant maps respectively, and  $\alpha \in \mathbb{R}$ . Then  $h_1 + \alpha h_2$  is  $(\tau_1 + \alpha \tau_2)$ -equivariant. In particular, the difference between two  $\tau$ -equivariant map is a  $\Gamma$ -invariant map.

**Fact 3.8.** If there are  $\tau, \tau' \in Z^1(\Gamma, \mathbb{R}^{d,1})$  such that there is a map  $\tau$  and  $\tau'$ -equivariant, then  $\tau = \tau'$ .

*Proof.* For any  $A \in \Gamma$  and any  $x$ , using the definition of equivariance, we obtain  $\langle A \cdot x, \tau(A) \rangle_d - \tau(A)_{d+1} = \langle A \cdot x, \tau'(A) \rangle_d - \tau'(A)_{d+1}$ .  $\square$

The following fact is clear from the definition of  $\tau$ -equivariant map and Lemma 2.29.

**Fact 3.9.** Let  $h : B^d \rightarrow \mathbb{R}$  be a  $C^2$   $\tau$ -equivariant function. Then  $L^{-1} \text{Hess } h$  is  $\Gamma$ -invariant:

$$(L^{-1} \text{Hess } h)(x)(X, Y) = (L^{-1} \text{Hess } h)(A \cdot x)(DA(x)(X), DA(x)(Y)) .$$

**Remark 3.10.** Fact 3.9 says that the second fundamental form of the hypersurface which is the graph of  $h$  (see Section 2.3.1) defines a symmetric  $(0, 2)$ -tensor on  $\mathbb{H}^d/\Gamma$ . Moreover this tensor is a symmetric Codazzi tensor, see Remark 2.32.

It can be useful to note the following converse to Fact 3.9.

**Lemma 3.11.** Let  $h : B^d \rightarrow \mathbb{R}$  be a  $C^2$  map such that  $L^{-1} \text{Hess } h$  is  $\Gamma$ -invariant. Then there exists a unique  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$  such that  $h$  is  $\tau$ -equivariant.

*Proof.* Let  $A \in \Gamma$ . As  $A$  acts as an affine map on  $B^d \times \mathbb{R}$ , by the rule of the Hessian of a composition (21) and the invariance of the Hessian, we obtain

$$\text{Hess}(h \circ A)(x)(X, Y) = \text{Hess } h(A \cdot x)(DA(x)(X), DA(x)(Y)) = \text{Hess } h(x)(X, Y) ,$$

hence  $h$  and  $h \circ A$  differ by an affine map, that in turn gives a vector  $\tau(A^{-1}) \in \mathbb{R}^{d,1}$ :

$$h(x) - h(A \cdot x) = \langle \tau(A^{-1}), \begin{pmatrix} x \\ 1 \end{pmatrix} \rangle_- .$$

Writing  $h(x) - h(A \cdot (B \cdot x))$  as  $h(x) - h(B \cdot x) + h(B \cdot x) - h(A \cdot (B \cdot x))$ , it follows that  $\tau$  satisfies the cocycle relation. Uniqueness is given by Fact 3.8.  $\square$

Now let us check that the discussion is not void. First there are easy examples in the coboundary case.

**Fact 3.12.** Let  $\tau_v$  be a coboundary, i.e. there is  $v \in \mathbb{R}^{d,1}$  such that  $\tau_v(A) = Av - v$ . Then  $h_v(x) = -\langle x, \bar{v} \rangle_d + v_{d+1}$  is a  $\tau_v$ -invariant map.

In full generality, if the cocycle is equal to zero, we know the function  $-L$  which is a  $C^\infty$   $\Gamma$ -invariant function with positive definite Hessian. By the very general ‘‘Ehresmann–Weil–Thurston holonomy principle’’ [Gol18], for cocycles close to 0 enough, there exist  $\tau$ -equivariant maps, which depends continuously on the cocycle. For convenience we recall the argument in our very simplified case, which follows the lines from [CEG06, Lemma I.1.7.2]. We need to take care about convexity, that is also classical [Gho02].

**Proposition 3.13.** For any cocycle  $\tau$  there exists a  $C^\infty$  convex (resp. concave)  $\tau$ -equivariant function  $h(\tau)$ .

Moreover, if  $\tau_n \rightarrow \tau$ , then there exist  $C^\infty$  convex (resp. concave)  $\tau_n$ -equivariant functions  $h(\tau_n)$  such that  $(h(\tau_n))_{n \in \mathbb{N}}$  converges to  $h(\tau)$ , and the second partial derivatives of  $h(\tau_n)$  converge to the second partial derivatives of  $h(\tau)$ .

*Proof.* Clearly it suffices to prove the statement for the convex case. Also by Fact 3.7, it suffices to prove it for any cocycle close to 0.

Let  $\{B_i(r_i)\}_{i=1,\dots,k}$  be disjoint open balls of  $\mathbb{H}^d$ , such that  $\Gamma \cdot \cup_i B_i(r_i)$  is a covering of  $\mathbb{H}^d$ . On  $B_1(r_1)$ , let us set  $h_1 = -L$ . For  $A \in \Gamma$  and  $y \in A \cdot B_1(r_1)$ , let us set  $h_1(y) = -L(y) + \langle \binom{y}{1}, \tau(A) \rangle_{d,1}$ . Such a function  $h_1$  is  $C^\infty$  and  $\tau$ -equivariant on  $\Gamma \cdot B_1(r_1)$ . The function  $h_1$  converge to  $-L$  uniformly on each orbit of  $B_1(r_1)$  if  $\tau$  goes to 0. Also the first partial derivatives of  $h_1$  converge to the ones of  $-L$  uniformly on each orbit of  $B_1(r_1)$  if  $\tau$  goes to 0. Moreover, the Hessian of  $h_1$  is equal to the one of  $-L$  on  $\Gamma \cdot B_1(r_1)$ , in particular it is positive definite.

Let  $r'_i < r_i$  for all  $i$ , such that  $\Gamma \cdot \cup_i B_i(r'_i)$  is still a covering of  $\mathbb{H}^d$ . Up to change the indices, suppose that  $B_2(r_2)$  has non empty intersection with the orbit of  $B_1(r_1)$ . Let  $W$  be an open neighborhood of  $B_2(r'_2) \cap \Gamma \cdot B_1(r'_1)$  such that its closure is contained in  $B_2(r_2) \cap \Gamma \cdot B_1(r_1)$ . Let  $\phi$  be a bump function which is equal to 1 on  $B_2(r'_2) \cap \Gamma \cdot B_1(r'_1)$  and whose support is contained in  $W$ . Note that the function  $\phi h_1$  is well-defined and  $C^\infty$  on  $\mathbb{H}^d$ , by setting the zero value out of  $W$ .

Let us define  $f = \phi h_1 + (1 - \phi)(-L)$  on  $B_2(r_2)$ . The function  $f$  is  $C^\infty$ , and equal to  $h_1$  on  $\Gamma \cdot B_1(r'_1) \cap B_2(r'_2)$ . When the cocycle goes to 0,  $f$  and its first and second derivatives go to  $-L$  and to its respective derivatives, uniformly on  $B_2(r'_2)$ . In particular, we suppose that the cocycle is sufficiently small, so that the Hessian of  $f$  is positive definite.

Then we define  $h_2 = f$  on  $B_2(r'_2)$ , and by equivariance we define  $h_2$  on  $\Gamma \cdot B_2(r'_2)$ . Also we set  $h_2 = h_1$  on  $\Gamma \cdot B_1(r'_1)$ . By construction,  $h_2$  is well defined on the non-empty intersections between orbits of  $B_1(r'_1)$  and orbits of  $B_2(r'_2)$ . Clearly,  $h_2$  converges to  $-L$  when the cocycle goes to 0. As the Hessian of  $h_2$  converges to the one of  $-L$  uniformly on  $B_2(r'_2)$ , by Fact 3.9, this is true on each element in the orbit of  $B_2(r'_2)$ , in particular the Hessian of  $h_2$  is positive definite.

In the same way, if  $r''_i < r'_i$  is such that  $\Gamma \cdot \cup_i B_i(r''_i)$  is still a covering of  $\mathbb{H}^d$ , then we can construct a function  $h_3$ , equivariant on the orbit of  $B_1(r''_1) \cup B_2(r''_2) \cup B_3(r''_3)$  and satisfying the statement of the proposition. After a finite number of steps, we have constructed the wanted functions. □

**Corollary 3.14.** *For any  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ , there exists a continuous map  $b_\tau : \partial B \rightarrow \mathbb{R}$  such that any  $\tau$ -equivariant map extends continuously as  $b_\tau$  on  $\partial B$ .*

*Moreover if  $\tau_1, \tau_2 \in Z^1(\Gamma, \mathbb{R}^{d,1})$  and  $\alpha \in \mathbb{R}$ , then  $b_{\alpha\tau_1 + \tau_2} = \alpha b_{\tau_1} + b_{\tau_2}$ . And  $\tau$  is a coboundary if and only if  $b_\tau$  is the restriction to  $\partial B^d$  of an affine map of  $\mathbb{R}^d$ .*

*Proof.* From Proposition 3.13, there exists a  $C^\infty$  convex  $\tau$ -equivariant map. From Fact 3.9,  $\mathbf{Mean}(h)$  is a  $\Gamma$ -invariant function, hence bounded, so by Proposition 2.34, there exists a continuous function  $b_\tau : \partial B \rightarrow \mathbb{R}$  that extends continuously  $h$ . As the difference of two  $\tau$ -equivariant map is a  $\Gamma$ -invariant function, and as a  $\Gamma$ -invariant function extends continuously as the zero function on  $\partial B$  (Fact 3.5), it follows that  $b_\tau$  is the continuous extension of any  $\tau$ -equivariant map.

The second property is obvious from the definition of  $b_\tau$  and Fact 3.7. The last property follows from Fact 3.12 □

From the existence of  $b_\tau$  we deduce easily the existence of a unique  $\tau$ -equivariant mean map in the following lemma. The maps whose graphs are the boundary of the convex hull of the graph of  $b_\tau$  will be introduced in Section 3.3.

**Corollary 3.15.** *Let  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ . There exists a unique  $C^\infty$   $\tau$ -equivariant map, denoted by  $h_\tau^{\text{mean}}$ , satisfying  $\mathbf{Mean}(h_\tau^{\text{mean}}) = 0$ . Moreover, for  $\alpha \in \mathbb{R}$  and  $\tau' \in Z^1(\Gamma, \mathbb{R}^{d,1})$ ,  $h_{\tau + \alpha\tau'}^{\text{mean}} = h_\tau^{\text{mean}} + \alpha h_{\tau'}^{\text{mean}}$ , and  $h_\tau^{\text{mean}}$  is the restriction to  $B^d$  of an affine map if and only if  $\tau$  is a coboundary.*

*Proof.* By Corollary 3.14 and Proposition 2.37, we know that there exists a unique  $C^\infty$  map, denoted by  $h_\tau^{\text{mean}}$ , having  $b_\tau$  as values on  $\partial B$ , and such that  $\mathbf{Mean}(h_\tau^{\text{mean}}) = 0$ . This map is  $\tau$ -equivariant. Indeed, apply an element of  $\Gamma_\tau$  to the graph of  $h_\tau^{\text{mean}}$ . Then we obtain the graph of a map with vanishing  $\mathbf{Mean}$  and boundary value  $b_\tau$ , so it has to be  $h_\tau^{\text{mean}}$  by uniqueness.

The second point is clear from Fact 3.7 and the fact that  $h_\tau^{\text{mean}}$  satisfies a linear equation, namely  $(\frac{1}{d}\Delta^{\mathbb{H}^d} - 1)h_\tau^{\text{mean}} = 0$ . The last point is immediate from Corollary 3.14.  $\square$

**Remark 3.16.** For any  $t \in \mathbb{R}$ , the map  $h_\tau^{\text{mean}} - tL$  is  $\tau$ -equivariant, with mean curvature equal to  $t$ . Hence the graphs of these maps gives a smooth foliation of  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$  by hypersurfaces of constant mean curvature.

Corollary 3.15 allows to recover a classical relation between cocycles and traceless Codazzi tensors [Laf83, OS83, BS16]. Let  $\text{Cod}_0^\Gamma$  be the vector space of traceless symmetric Codazzi tensors on  $\mathbb{H}^d/\Gamma$ . Let  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ . By Corollary 3.15, there is a map  $h_\tau^{\text{mean}}$  whose second fundamental form is a  $\Gamma$ -invariant traceless Codazzi tensor (see Remark 3.10), hence it defines an element of  $\text{Cod}_0^\Gamma$ , denoted by  $\mathbf{Cod}(\tau)$ . By Corollary 3.15, the map  $\mathbf{Cod} : Z^1(\Gamma, \mathbb{R}^{d,1}) \rightarrow \text{Cod}_0^\Gamma$  is linear. The kernel of this map corresponds to the  $\tau$  such that  $h_\tau^{\text{mean}}$  is affine, hence to the coboundaries by Corollary 3.15. We thus obtain an injective morphism from  $H^1(\Gamma, \mathbb{R}^{d,1})$  to  $\text{Cod}_0^\Gamma$ , still denoted by  $\mathbf{Cod}$ .

**Proposition 3.17.** *The map  $\mathbf{Cod} : H^1(\Gamma, \mathbb{R}^{d,1}) \rightarrow \text{Cod}_0^\Gamma$  is an isomorphism.*

*Proof.* Let  $C \in \text{Cod}_0^\Gamma$ , that defines a  $\Gamma$ -invariant symmetric traceless Codazzi tensor  $\tilde{C}$  on  $\mathbb{H}^d$ . By Lemma 2.13, there exists  $h : B^d \rightarrow \mathbb{R}$  such that  $\tilde{C} = \mathbb{I}_h$ . From Lemma 3.11, there exists a cocycle  $\tau$  such that  $h$  is  $\tau$ -equivariant, hence as  $\mathbb{I}_h$  is traceless, by the uniqueness part of Corollary 3.15, we will have  $h = h_\tau^{\text{mean}}$ .  $\square$

### 3.3 Volume of the convex core and asymmetric norm

#### 3.3.1 Convex core

Let  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ . There is an associated map  $b_\tau : \partial B^d \rightarrow \mathbb{R}$  given by Corollary 3.14. This map has a graph  $\Lambda(b_\tau)$ , and we will look at its convex hull  $\text{CH}(\tau)$  in the affine space  $\mathbb{R}^{d+2}$ , as well as the functions  $h_{b_\tau}^-$  and  $h_{b_\tau}^+$  (see Section 2.3.3) whose graphs are the boundary of  $\text{CH}(\tau)$ . We will denote those two last maps by  $h_\tau^-$  and  $h_\tau^+$  respectively.

The argument to check the following fact is analogous to the one used in the proof of Corollary 3.15.

**Fact 3.18.** *The map  $h_\tau^-$  and  $h_\tau^+$  are  $\tau$ -equivariant, in particular  $\text{CH}(\tau)$  is globally invariant for the action of  $\Gamma_\tau$ .*

**Lemma 3.19.** *Let  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ . Then:*

1. for any convex (resp. concave)  $\tau$ -equivariant map  $h$ , then  $h \leq h_\tau^-$  (resp.  $h \geq h_\tau^+$ ),
2.  $h_\tau^+ = -h_{-\tau}^-$ ,
3. For  $\alpha > 0$ ,  $h_{\alpha\tau}^- = \alpha h_\tau^-$ ,
4.  $h_\tau^- + h_{\tau'}^- \leq h_{\tau+\tau'}^-$  and  $h_\tau^+ + h_{\tau'}^+ \geq h_{\tau+\tau'}^+$ .

*Proof.* The two first points are from the definitions of  $h_\tau^+$  and  $h_\tau^-$ , Proposition 2.40 and Corollary 3.14. The third point follows from (30) and the fact that  $b_{\alpha\tau} = \alpha b_\tau$ . The fourth point follows from the first point, as  $h_\tau^- + h_{\tau'}^-$  is a convex  $(\tau + \tau')$ -equivariant function.  $\square$

**Lemma 3.20.** *Let  $\tau_v$  be a coboundary, then  $h_{\tau_v}^- = h_{\tau_v}^{\text{mean}}$  is an affine map and  $h_{\tau+\tau_v}^- = h_\tau^- + h_{\tau_v}^-$ . Conversely, if  $h_\tau^- = h_\tau^{\text{mean}}$ , then  $h_\tau^{\text{mean}}$  is affine and  $\tau$  is a coboundary.*

*Proof.* If  $\tau$  is a coboundary, we know that there exists a  $\tau$ -equivariant affine map (Fact 3.12). Hence the convex hull of  $\Lambda(b_\tau)$  is a piece of an hyperplane, and this hyperplane is also the  $\tau_v$ -mean hypersurface. Then  $h_{\tau+\tau_v}^- = h_\tau^- + h_{\tau_v}^-$  follows from (30) because  $h_{\tau_v}$  is an affine map. For the second part, on the one hand,  $\mathbf{Mean}(h_\tau^{\text{mean}}) = 0$ . On the other hand, if  $h_\tau^{\text{mean}} = h_\tau^-$ , then  $h_\tau^{\text{mean}}$  is convex, hence affine (Remark 2.33), so  $b_\tau$  is the restriction to  $\partial B^d$  of an affine map. By Corollary 3.14,  $\tau$  is a coboundary.  $\square$

**Definition 3.21.** *The convex core of  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$ , denoted by  $\text{CC}(\tau)$ , is the smallest non-empty convex set of  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$ .*

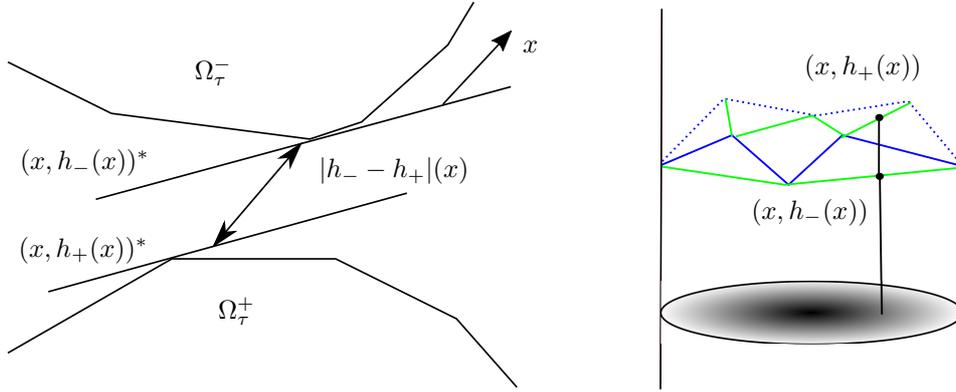


Figure 8: The volume of the convex core  $\text{CC}(\tau)$  is a “mean distance” between a past complete flat GHMC spacetime and a future complete flat GHMC spacetime with the same holonomy.

In the above definition, “convex” has to be understood in the strong sense of geodesically convex:  $C$  is convex if for  $x, y \in C$ , any geodesic between  $x$  and  $y$  belongs to  $C$ . So for example, a single point or a small open ball may not be convex. In the cylindrical model of the universal cover, this notion of convexity coincides with the affine one.

Clearly,  $\text{CC}(\tau) = \text{CH}(\tau)/\Gamma_\tau$ . Hence  ${}^*\mathbb{R}^{d,1}/\Gamma_\tau$  has a compact convex core, so the action of  $\Gamma_\tau$  on  ${}^*\mathbb{R}^{d,1}$  is *convex cocompact*, in the sense of [DGK17b, DGK17a].

Recall the volume form on co-Minkowski space, Section 2.2.6. Let us denote by  $\text{Vol}$  the induced volume on  ${}^*\mathbb{R}^{d,1}$ . It is then immediate that for any  $\tau \in Z^1(\Gamma, \mathbb{R}^{d,1})$ ,

$$\text{Vol}(\text{CC}(\tau)) = \int_{\mathbb{H}^d/\Gamma} h_\tau^+ - h_\tau^- . \quad (37)$$

Here by abuse of notation, we denote in the same way the  $\Gamma$ -invariant function  $h_\tau^+ - h_\tau^-$  and the corresponding function on the compact hyperbolic manifold  $\mathbb{H}^d/\Gamma$ . The integration is implicitly with respect to the volume form given by the hyperbolic metric.

**Definition 3.22.** *The function  $\mathbf{vol} : H^1(\Gamma, \mathbb{R}^{d,1}) \rightarrow \mathbb{R}$  associates  $\text{Vol}(\text{CC}(\tau))$  to any representative  $\tau$  of an element of  $H^1(\Gamma, \mathbb{R}^{d,1})$ .*

By Lemma 3.20,  $\mathbf{vol}$  is well-defined. Actually, the following result is straightforward to check from Lemma 3.20 and Lemma 3.19.

**Proposition 3.23.**  *$\mathbf{vol}$  is a norm on  $H^1(\Gamma, \mathbb{R}^{d,1})$ .*

**Remark 3.24.** The volume of the convex core has the following geometric meaning in Minkowski space. From a cocycle  $\tau$ , we have the boundary map  $b_\tau$ , that defines two convex sets  $\Omega_{b_\tau}^+$  and  $\Omega_{b_\tau}^-$  in Minkowski space, see Remark 2.44. It follows from the previous section that those two sets (here denoted by  $\Omega_\tau^+$  and  $\Omega_\tau^-$ ), are invariant under the action of  $\Gamma_\tau$  on Minkowski space. Actually the action is free and properly discontinuous on  $\Omega_\tau^+ \cup \Omega_\tau^-$ , and the quotient  $\Omega_\tau^-$  (resp.  $\Omega_\tau^+$ ) is a *future complete flat* (resp. *past complete*) Globally Hyperbolic Maximal Cauchy Compact (in short, GHMC) spacetime. As the addition of a coboundary to the cocycle  $\tau$  will only change the origin in Minkowski space, then  $H^1(\Gamma, \mathbb{R}^{d,1})$  parametrizes the space of future complete (or past complete) flat GHMC spacetimes with a given linear holonomy, up to conjugacy. See [Bar05, Bon05] for more details.

Moreover, for any cocycle  $\tau$ , we have that  $-\Omega_\tau^- = \Omega_{-\tau}^+$ . Now, for any  $x \in B^d$ , let us denote by  $\mathbf{width}(x)$  the Lorentzian distance between the support plane of  $\Omega_\tau^+$  with outward unit normal  $\begin{pmatrix} x \\ 1 \end{pmatrix}$ , and the support plane of  $\Omega_\tau^-$  with inward unit normal  $\begin{pmatrix} x \\ 1 \end{pmatrix}$ . Note that the map  $x \mapsto \mathbf{width}(x)$  is  $\Gamma$ -invariant. Then the mean width, defined as  $\int_{\mathbb{H}^d/\Gamma} \mathbf{width}(\cdot)$ , is given by (37), see Figure 8.

### 3.3.2 Asymmetric norm

In the previous section we showed that the volume of the convex core is a norm on  $H^1(\Gamma, \mathbb{R})$ . We now see that it is actually the symmetrization of an asymmetric norm on  $H^1(\Gamma, \mathbb{R})$ .

For a cocycle  $\tau$ , the  $S_1$  norm is defined as follows

$$\|\tau\|_{S^1} = \int_{\mathbb{H}^d/\Gamma} h_\tau^{\text{mean}} - h_\tau^- . \quad (38)$$

(The denomination will be motivated in Remark 3.26.)

By Lemma 3.20, if  $\tau_v$  is a coboundary, then  $\|\tau + \tau_v\| = \|\tau\| + \|\tau_v\| = \|\tau\|$ . Hence  $\|\cdot\|$  is well defined on  $H^1(\Gamma, \mathbb{R}^{d+1})$ .

**Proposition 3.25.** *The  $S_1$  norm  $\|\cdot\|_{S_1}$  defines an asymmetric norm on  $H^1(\Gamma, \mathbb{R}^{d+1})$ , i.e.  $\forall [\tau], [\tau'] \in H^1(\Gamma, \mathbb{R}^{d+1})$*

1.  $\|[\tau]\|_{S_1} \geq 0$ ;
2.  $\|[\tau]\|_{S_1} = 0$  if and only if  $[\tau] = 0$ .
3.  $\|[\tau] + [\tau']\|_{S_1} \leq \|[\tau]\|_{S_1} + \|[\tau']\|_{S_1}$ .
4.  $\forall \alpha \geq 0, \|\alpha[\tau]\|_{S_1} = \alpha\|[\tau]\|_{S_1}$ ;

*Proof.* The first property comes from Lemma 2.42. The second point is Lemma 3.20. The third and fourth points are immediate consequence of Lemma 3.19.  $\square$

It is obvious from (38) and (37) that  $\mathbf{vol}$  the symmetrization of  $\|\cdot\|_{S_1}$ :

$$\mathbf{vol}([\tau]) = \frac{1}{2} (\|[\tau]\|_{S_1} + \| - [\tau] \|_{S_1}) .$$

### 3.3.3 Mean curvature measure

We now explain how  $\|\cdot\|_{S^1}$  is related to the mean curvature measure introduced in Section 2.3.4. From Lemma 2.47, we have that for any convex  $\tau$ -equivariant map  $h$ , the measure  $\mathbf{MM}(h)$  is  $\Gamma$ -invariant, and then defines a Radon measure  $\mathbf{MM}^\Gamma(h)$  on  $\mathbb{H}^d/\Gamma$ . Actually, there is a nice expression for this measure. Let  $h$  be a convex  $\tau$ -equivariant function. By definition of  $h_\tau^{\text{mean}}$ ,  $\mathbf{MM}(h) = \mathbf{MM}(h) - \mathbf{MM}(h_\tau^{\text{mean}}) = \mathbf{MM}(h - h_\tau^{\text{mean}})$ . On the other hand,  $h - h_\tau^{\text{mean}}$  is  $\Gamma$ -invariant, so we deduce easily that for any  $C^\infty$  function  $\varphi$  on  $\mathbb{H}^d/\Gamma$ ,

$$\mathbf{MM}^\Gamma(h)(\varphi) = \int_{\mathbb{H}^d/\Gamma} (h - h_\tau^{\text{mean}}) \left( \frac{1}{d} \Delta^{\mathbb{H}^d} - 1 \right) \varphi .$$

Taking  $\varphi = 1$ ,

$$\mathbf{MM}^\Gamma(h)(\mathbb{H}^d/\Gamma) = \int_{\mathbb{H}^d/\Gamma} h_\tau^{\text{mean}} - h ,$$

in particular, if  $h = h_\tau^-$ , by definition of the  $S_1$  norm,

$$\|\tau\|_{S^1} = \mathbf{MM}^\Gamma(h_\tau^-)(\mathbb{H}^d/\Gamma) . \quad (39)$$

**Remark 3.26.** Consider the convex set  $\Omega_\tau^-$  in Minkowski space, as well as the  $\epsilon$ -equidistant convex set  $\Omega_\tau^-(\epsilon)$  (it is the dual convex set in Minkowski space of the epigraph of  $h_\tau^- - \epsilon$  in  $B^d \times \mathbb{R}$ ). By a Lorentzian version of the Steiner formula proved in [FV16], the volume of  $(\Omega_\tau^- \setminus \Omega_\tau^-(\epsilon))/\Gamma_\tau$  is a polynomial in  $\epsilon$  of degree  $d + 1$ . Up to a dimensional constant, the coefficient in front of  $\epsilon^d$  is nothing but  $\mathbf{MM}^\Gamma(h_\tau^-)(\mathbb{H}^d/\Gamma)$ . The analogous quantity in the classical theory of convex bodies is called the (total) area measure of order one [Sch14], and usually denoted by  $S_1$ , that explains our terminology (see also Remark 2.55).

**Lemma 3.27.** *Let  $\tau_n \rightarrow \tau$ . Then  $b_{\tau_n}$  (resp.  $h_{\tau_n}^-$ ) pointwise converge to  $b_\tau$  (resp.  $h_\tau^-$ ).*

*Proof.* By Proposition 3.13, we have convex (resp. concave)  $\tau_n$ -equivariant functions converging to a  $\tau$ -equivariant convex (resp. concave) function. For any  $n$ , as the concave and the convex  $\tau_n$ -equivariant functions coincide on  $\partial B^d$  with  $b_{\tau_n}$  given by Corollary 3.14, they bound a convex body  $K_n$  of  $\mathbb{R}^{d+1}$ . Let us denote by  $K$  the convex body bounded by the  $\tau$ -equivariant convex and concave functions.

Let us denote by  $C^{d+1}$  the space of non-empty compact sets of  $\mathbb{R}^{d+1}$ , endowed with the Hausdorff topology. Suppose that there is a subsequence  $(K_{n_i})$  of  $(K_n)$  that converges to  $K'$  in  $C^{d+1}$ . Then  $K'$  is a convex body [Sch14, Theorem 1.8.6]. Moreover, each point of  $K'$  is the limit of a sequence of points  $(x_{n_i})$  with  $x_{n_i} \in K_{n_i}$  [Sch14, Theorem 1.8.8]. From this it is easy to deduce that  $K' = K$ .

Now as the  $\tau_n$ -equivariant functions are converging, they are bounded, and in turn the sequence of convex bodies  $(K_n)_n$  is bounded in  $B^d \times \mathbb{R} \subset \mathbb{R}^{d+1}$ . By the Blaschke selection theorem [Sch14, Theorem 1.8.7], there is a subsequence  $K_{n_i}$  converging to a convex body  $K'$ . Moreover, the sequence  $K_n$  is contained in a compact subspace of  $C^{d+1}$  [Sch14, Theorem 1.8.4]. As we saw that any convergent subsequence of  $(K_n)$  converges to  $K$ , it follows that  $(K_n)$  converges to  $K$ .

As the limit of any convergent sequence  $(x_{n_i})$  with  $x_{n_i} \in K_{n_i}$  must belong to  $K$  [Sch14, Theorem 1.8.8], it is easy to deduce that  $b_{\tau_n} \rightarrow b_\tau$ . This easily implies the Hausdorff convergence of  $\text{CH}(\tau_n)$  to  $\text{CH}(\tau)$ , see e.g. [Smi, Lemma 2.1], which in turn gives the convergence of  $h_{\tau_n}^-$  to  $h_\tau^-$ , as the Hausdorff convergence of convex bodies implies the Hausdorff convergence of the boundaries [Sch14, Lemma 1.8.1].  $\square$

**Remark 3.28.** By standard properties of convex functions, it follows from Lemma 3.27 that  $h_{\tau_n}^-$  converges to  $h_\tau^-$  uniformly on any compact sets of  $B^d$ . But one cannot deduce a uniform convergence of  $b_{\tau_n}$  from the pointwise convergence [hm].

Actually we will obtain the uniform convergence of the  $h_{\tau_n}^-$  as a byproduct of the considerations of Section 4. In particular, Lemma 4.11 will imply the following proposition, without mention to the mean curvature measure.

**Proposition 3.29.** *The  $S_1$  norm  $\|\cdot\|_{S_1} : H^1(\Gamma, \mathbb{R}^{d,1}) \rightarrow \mathbb{R}$  is continuous.*

*Proof.* Let  $\tau_n \rightarrow \tau$ . From Lemma 3.27,  $h_{\tau_n}^-$  converges to  $h_\tau^-$ , and, using a partition of the unity, it is not hard to deduce from Lemma 2.46 that  $\mathbf{MM}^\Gamma(h_{\tau_n}^-)$  weakly converge to  $\mathbf{MM}^\Gamma(h_\tau^-)$ , so that the result follows from (39).  $\square$

### 3.3.4 Simplicial measured geodesic laminations

We use the notations and definitions of Section 3.1, where we have considered a simplicial measured geodesic lamination  $\lambda$  on the compact hyperbolic manifold  $\mathbb{H}^d/\Gamma$ . Namely we have supposed that  $\mathbb{H}^d/\Gamma$  contains  $n$  disjoint embedded totally geodesic hypersurfaces  $H_1, \dots, H_n$  with positive weights  $\omega_i$ .

Let us push the construction a step forward. For any  $y \in B^d$ , let  $c : [0, 1] \rightarrow B^d$  be any curve transverse to  $\tilde{L}$  joining the base point  $\tilde{x}$  to  $y$ , and define

$$h_\lambda(y) = \sum_{j \in c([0,1]) \cap \tilde{L}} \mu_j h_{l_j}(y)$$

where  $h_{l_j}$  is the canonical map associated with  $l_j$  (Definition 2.48).

**Fact 3.30.** *If  $\tau_\lambda$  is the cocycle given by (36), then  $h_\lambda = h_{\tau_\lambda}^-$ .*

*Proof.* As the weights are positive, by Fact 2.50,  $h_\lambda$  is a convex map.

Let us check that  $h_\lambda$  is  $\tau_\lambda$ -equivariant. Let  $\tilde{c} : [0, 1] \rightarrow B^d$  be a path joining  $\tilde{x}$  to  $A \cdot \tilde{x}$ , and let  $c' : [0, 1] \rightarrow B^d$  be a path joining  $\tilde{x}$  to  $y$ , both assumed to be transverse to  $\tilde{L}$ . Then the concatenation of  $c$  with  $A \cdot c'$  is a path joining  $\tilde{x}$  to  $A \cdot y$ , hence

$$h_\lambda(A \cdot y) = \sum_{j \in c([0,1]) \cap \tilde{L}} \omega_j h_{l_j}(y) + \sum_{j \in (A \cdot c')([0,1]) \cap \tilde{L}} \omega_j h_{l_j}(y),$$

and as by definition  $h_l(y) = \langle \binom{y}{1}, v_l \rangle_{d,1}$ , then  $\sum_{j \in (c([0,1]) \cap \tilde{L})} \omega_j h_{l_j}(y) = \langle \binom{y}{1}, \tau_\lambda(A) \rangle_{d,1}$ . Also,  $\sum_{j \in (A \cdot c'([0,1]) \cap \tilde{L})} \omega_j h_{l_j}(y) = \sum_{j \in (c'([0,1]) \cap \tilde{L})} \omega_j h_{A \cdot l_j}(A \cdot y)$ , and by Fact 2.52,  $h_{A \cdot l_j}(A \cdot y) = \frac{L(A \cdot y)}{L(y)} h_l(y)$ . The equivariance is proved.

A  $h_\lambda$  is a convex  $\tau_\gamma$ -equivariant map, then  $h_\lambda \leq h_{\tau_\lambda}^-$ . By construction, the graph of  $h_\lambda$  is made of segments joining points of graph of  $b_{\tau_\lambda}$ , hence it is contained in  $\text{CH}(b_{\tau_\lambda})$ , so  $h_\lambda \geq h_{\tau_\lambda}^-$ .  $\square$

The *length*  $\mathbf{length}(\lambda)$  of a simplicial measured geodesic lamination  $\lambda$  on  $\mathbb{H}^d/\Gamma$  is defined as sum of the weights times the total volume of the corresponding totally geodesic hypersurfaces. By Lemma 2.54 and (39), we obtain the following.

**Proposition 3.31.** *Let  $\lambda$  be a simplicial measured geodesic lamination on  $\mathbb{H}^d/\Gamma$ . Then*

$$\mathbf{length}(\lambda) = \|\tau_\lambda\|_{S^1} .$$

**Remark 3.32.** There is no reason why for  $d \geq 3$  any cocycle should arise from a (simplicial) measured geodesic lamination on  $\mathbb{H}^d/\Gamma$ . So for  $d \geq 3$ , the concept of measured geodesic lamination is not sufficient. A more suitable concept is the one of *measured geodesic stratification*, introduced in [Bon05]. In contrast, we will see in the next section that for  $d = 2$ , any cocycle arises from a measured geodesic lamination.

### 3.4 The case of dimension 2 + 1

In this part we study the particularities of the  $d = 2$  case. We will denote by  $\text{Teich}_S$  the Teichmüller space of a compact surface homeomorphic to  $\mathbb{H}^2/\Gamma$ . We will denote by  $g$  the genus of  $S$ .

#### 3.4.1 Goldman isomorphism

The Teichmüller space  $\text{Teich}_S$  can be defined as the space of faithful and discrete representations of  $\pi_1 S$  into  $\text{Isom}_0(\mathbb{H}^2)$  up to conjugacy. Let  $\rho$  be such a representation such that  $\Gamma = \rho(\pi_1 S)$ . Then the tangent space of  $\text{Teich}_S$  at  $\rho$  naturally identifies with  $H^1(\pi_1(S), \mathbf{isom}(\mathbb{H}^2))$ , where  $\pi_1 S$  acts on the Lie group  $\mathbf{isom}(\mathbb{H}^2)$  via  $\text{Ad } \rho$  [Gol84]. Using the hyperboloid model  $\mathcal{H}^2$  for  $\mathbb{H}^2$ ,  $\mathbf{isom}(\mathbb{H}^2)$  can be identified with  $\mathfrak{o}(2, 1)$ . Let us write it as follows.

**Theorem 3.33** ([Gol84]). *There is a vector space isomorphism*

$$\mathbf{Gold} : H^1(\Gamma, \mathfrak{o}(2, 1)) \rightarrow T_{\mathbb{H}^2/\Gamma} \text{Teich}_S .$$

There is also a one-to-one correspondence between vectors of  $\mathbb{R}^{2,1}$  and infinitesimal Minkowski isometries of  $\mathfrak{o}(2, 1)$  —this may be seen for example using the Minkowski cross product, see e.g. [DG99]. This identification gives a vector space isomorphism

$$\mathbf{C} : H^1(\Gamma, \mathbb{R}^{2,1}) \rightarrow H^1(\Gamma, \mathfrak{o}(2, 1)) ,$$

and in turn we have the following vector space isomorphism

$$\xi = \mathbf{Gold} \circ \mathbf{C} : H^1(\Gamma, \mathbb{R}^{2,1}) \rightarrow T_{\mathbb{H}^2/\Gamma} \text{Teich}_S .$$

In particular, we obtain the following.

**Corollary 3.34.** *The vector space  $H^1(\Gamma, \mathbb{R}^{2,1})$  has dimension  $6g - 6$ .*

#### 3.4.2 Mess homeomorphism

Let us call an *entire segment* of  $B^2$  a segment of  $B^2$  whose endpoints are in  $\partial B^2$ . A *geodesic lamination*  $\tilde{L}$  of  $B^2$  is a non-empty closed union of disjoint entire segments of  $B^2$ . Let  $\tilde{L}$  be a geodesic lamination on  $B^2$  which is invariant under the action of  $\Gamma$ . Then the image  $L$  of  $\tilde{L}$  under the projection is a geodesic lamination on the compact hyperbolic surface  $B^2/\Gamma$ . A *measured geodesic lamination*  $\lambda = (L, \mu)$  on  $B^2/\Gamma$  is the data of a geodesic lamination  $L$  together with a *transverse measure*  $\mu$ , that is the data of a Radon measure on each compact rectifiable curve transverse to  $L$ , such that

- the support of the measure is the intersection of the arc with  $L$ ,
- if two arcs are homotopic through arc transverse to  $L$ , then the homotopy sends the measure on one segment to the measure on the other one.

A simplicial measured geodesic lamination on  $B^2/\Gamma$  is a set of non-intersecting closed simple geodesics weighted by positive numbers. Note that the action of  $\Gamma$  onto  $B^2$  is via the identification of the disc with the Klein model of the hyperbolic plane, but the notation  $B^2$  stands for reminding the affine nature of measured geodesic lamination on the universal cover.

Let  $\mathcal{ML}_\Gamma$  be the set of measured geodesic laminations on the compact hyperbolic surface  $\mathbb{H}^2/\Gamma$ .  $\mathcal{ML}_\Gamma$  is endowed with the following topology. We say that  $\lambda_n$  converge to  $\lambda$  if, for any compact segment  $c$  transverse to  $L$  then

- $c$  is transverse to  $L_n$  for  $n$  big,
- $\mu_n$  weakly converge to  $\mu$  on  $c$ .

We have the following classical result of Thurston, see e.g. [Bon01] and the references therein.

**Theorem 3.35** (Thurston). *For the topology defined above,  $\mathcal{ML}_\Gamma$  is a manifold of dimension  $6g - 6$ .*

Recall from (34) that a vector  $v_l$  of  $\mathbb{R}^{2,1}$  is assigned to any entire segment  $l$  of  $B^2$ . Let  $\mathbf{e}$  be a continuous function such that, for any path  $c : [0, 1] \rightarrow B^2$  transverse to  $L$ ,  $\mathbf{e}_L(c(t)) = v_l$  is  $c(t) \in l$  and  $l \in \tilde{L}$ . Let us fix an arbitrary base point  $\tilde{x} \in B^2$ . Then define, for  $A \in \Gamma$ , and for any path  $c : [0, 1] \rightarrow B^2$  transverse to  $L$  joining  $\tilde{x}$  and  $A \cdot \tilde{x}$ :

$$\tau_\lambda(A) = \int_0^1 \mathbf{e}_L(c(t)) d\mu(t). \quad (40)$$

As the measure is transverse, the definition of  $\tau_\lambda$  is independent from the choice of the path  $c$  and the function  $\mathbf{e}_L$ . The following fact is proved formally in the same way as Facts 3.2 and 3.3.

**Fact 3.36.** *We have  $\tau_\lambda \in Z^1(\Gamma, \mathbb{R}^{2,1})$ . Moreover, if the basepoint is changed, the new cocycle differ from the preceding one by a coboundary.*

Hence we have constructed a well defined map

$$\mathbf{Mess} : \mathcal{ML}_\Gamma \rightarrow H^1(\Gamma, \mathbb{R}^{d,1}).$$

**Theorem 3.37** ([Mes07]). *The map  $\mathbf{Mess}$  defined above is a homeomorphism.*

*Proof.* The map is clearly injective and continuous. By Theorem 3.35 and Corollary 3.34, both  $\mathcal{ML}_\Gamma$  and  $H^1(\Gamma, \mathbb{R}^{d,1})$  are manifolds of same dimension. Hence  $\mathbf{Mess}$  is a local homeomorphism by the invariance of domain theorem. Now for  $\lambda \in \mathcal{ML}_\Gamma$  and  $t \geq 0$ , let us define  $t\lambda$  as the measured geodesic lamination obtained from  $\lambda$  by simply multiplying the transverse measure by  $t$ . By (36) we clearly have  $\mathbf{Mess}(t\lambda) = t\mathbf{Mess}(\lambda)$ . As  $H^1(\Gamma, \mathbb{R}^{2,1})$  is a vector space and  $\mathbf{Mess}$  a local homeomorphism, it follows that  $\mathbf{Mess}$  is surjective.  $\square$

**Remark 3.38.** We could have given a direct proof of Theorem 3.37, by defining a “bending measure” belonging to  $\mathcal{ML}_\Gamma$  from the graph of  $h_\tau^-$ , for any cocycle  $\tau$ . There are at least three ways to define such a bending measure. The first one is to mimic the construction of the bending measure given by the upper boundary component of the convex core of a hyperbolic quasi-Fuchsian manifolds [EM06]. The second one is to define, as in [Mes07], the induced distance on the spacelike part of the boundary of  $\Omega_\tau^-$ , the dual of the epigraph of  $h_\tau^-$  in Minkowski space. The last one is to consider the mean curvature measure given by  $h_\tau^-$ .

### 3.4.3 Length of measured geodesic laminations

We have encountered the length of simplicial measured geodesic laminations in Section 3.3.4. For  $d = 2$ , the length of a measured geodesic lamination is defined as the total mass on the

surface of the measure which is the product of the hyperbolic measure along the leaves of the lamination and the measure transverse to the leaves. We refer to [Bon01] for more details. Actually, the simplicial case suffices, as the following results shows. One may see for example Lemma 2.4 in [Ker85] for the first one, and Theorem 3.1.3 in [PH92] or Section 3.4.3 in [BB09] for the second one.

**Proposition 3.39.** *The map  $\mathbf{length} : \mathcal{ML}_\Gamma \rightarrow \mathbb{R}$  is continuous.*

**Proposition 3.40.** *Simplicial measured geodesic laminations are dense in  $\mathcal{ML}_\Gamma$ .*

So from the above results, Proposition 3.31 generalizes as follows.

**Proposition 3.41.** *Let  $\lambda \in \mathcal{ML}_\Gamma$ . Then*

$$\mathbf{length}(\lambda) = \|\mathbf{Mess}(\lambda)\|_{S^1} .$$

### 3.4.4 Thurston earthquake norm

From a measured geodesic lamination  $\lambda$  on  $\mathbb{H}^2/\Gamma$ , one obtains another hyperbolic metric on  $S$  by performing a (left) earthquake along the lamination —we refer to [Ker85] and the reference therein for more details about earthquakes. Actually for  $t$  near 0, earthquakes along  $t\lambda$  define a path in  $\mathrm{Teich}_S$  starting at  $\mathbb{H}^2/\Gamma$ . This path has a well defined derivative at 0, which gives an element in  $T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S$ , the tangent space of Teichmüller space at the point  $\mathbb{H}^2/\Gamma$ . In turn, we have an *infinitesimal earthquake map*:

$$\mathbf{InfEarth} : \mathcal{ML}_\Gamma \rightarrow T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S .$$

**Theorem 3.42** ([Ker85, Proposition 2.6]). *The map  $\mathbf{InfEarth}$  is a homeomorphism.*

So the map  $\mathbf{InfEarth} \circ \mathbf{Mess}^{-1}$  provides a homeomorphism between  $H^1(\Gamma, \mathbb{R}^{2,1})$  and  $T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S$ . Although there is no natural vector space structure on  $\mathcal{ML}_\Gamma$ , we have the following.

**Proposition 3.43** ([BS12, Proposition B.3]). *We have that  $\mathbf{InfEarth} \circ \mathbf{Mess}^{-1} = \xi$ . In particular,  $\mathbf{InfEarth} \circ \mathbf{Mess}^{-1}$  is a vector space isomorphism.*

In other terms, as  $\xi = \mathbf{Gold} \circ \mathbf{C}$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{ML}_\Gamma & \xrightarrow{\mathbf{InfEarth}} & T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S \\ \mathbf{Mess} \downarrow & & \uparrow \mathbf{Gold} \\ H^1(\Gamma, \mathbb{R}^{2,1}) & \xrightarrow{\mathbf{C}} & H^1(\Gamma, \mathfrak{o}(2,1)) \end{array} .$$

**Definition 3.44.** *Let  $X \in T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S$ . The earthquake norm of  $X$  is*

$$\|X\|_{\mathrm{earth}} = \mathbf{length}(\mathbf{InfEarth}^{-1}(X)) .$$

From Proposition 3.41 and Proposition 3.43, one has in fact

$$\|X\|_{\mathrm{earth}} = \|\xi^{-1}(X)\|_{S^1}$$

and as  $\xi$  is a vector space isomorphism, from Proposition 3.25, one finally obtains the following result.

**Theorem 3.45** ([Thu98, Theorem 5.2]). *The earthquake norm is an asymmetric norm on  $T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S$ .*

**Remark 3.46.** There is a smooth analogue of Proposition 3.43 proved in [BS16]. Namely, Proposition 3.17 gives a map  $\mathbf{Cod}$  from  $H^1(\Gamma, \mathbb{R}^{d,1})$  to  $\mathrm{Cod}_0^\Gamma$ , the space of traceless symmetric Codazzi tensors on  $\mathbb{H}^2/\Gamma$ . In dimension 2, there is also an isomorphism  $\mathbf{InfDef}$  from  $\mathrm{Cod}_0^\Gamma$  to  $T_{\mathbb{H}^2/\Gamma} \mathrm{Teich}_S$ , where a  $(0, 2)$ -tensor is seen as an infinitesimal deformations of the hyperbolic

metric (see [Tro92], where such tensors are called TT, for *transverse traceless*). Then, if  $J$  is the almost complex structure of  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$ , the following diagram commutes:

$$\begin{array}{ccc} \text{Cod}_0^\Gamma & \xrightarrow{\text{InfDef}} & T_{\mathbb{H}^2/\Gamma} \text{Teich}_S \\ \text{Cod}^{-1} \downarrow & & \downarrow J \\ H^1(\Gamma, \mathbb{R}^{2,1}) & \xrightarrow{\xi} & T_{\mathbb{H}^2/\Gamma} \text{Teich}_S \end{array} .$$

### 3.4.5 Thurston length norm

Following [Thu98], we note that two successive identifications of the tangent space  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$  of Teichmüller space with the cotangent space  $T_{\mathbb{H}^2/\Gamma}^* \text{Teich}_S$  will permit to define another asymmetric norm on  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$ , that is actually the Thurston length norm (see the Introduction).

A first identification between  $T_{\mathbb{H}^2/\Gamma}^* \text{Teich}_S$  and  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$  is given by the Weil–Petersson form of Teichmüller space, that is a symplectic form on  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$ . For  $\alpha \in T_{\mathbb{H}^2/\Gamma}^* \text{Teich}_S$ , let  $\alpha^\sharp$  be the dual element in  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$  of  $\alpha$  for the symplectic form. Then define naturally

$$\|\alpha\|_{\text{length}}^* := \|\alpha^\sharp\|_{\text{earth}} .$$

On the other hand, for any vector space  $E$  endowed with an asymmetric norm  $N$ , then its dual  $E^*$  is endowed with the asymmetric norm  $N^*$  defined by, for  $v \in E^*$ ,

$$N^*(v) := \sup \left\{ \frac{v(x)}{N(x)} \mid x \in E \setminus \{0\} \right\} .$$

Applying this to the cotangent space of Teichmüller space endowed with  $\|\cdot\|_{\text{length}}^*$ , we obtain a new asymmetric norm on the tangent space of Teichmüller space.

**Definition 3.47.** *Let  $X \in T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$ . The length norm of  $X$  is*

$$\|X\|_{\text{length}} = \sup \left\{ \frac{\alpha(X)}{\|\alpha^\sharp\|_{\text{earth}}} \mid \alpha \in T_{\mathbb{H}^2/\Gamma}^* \text{Teich}_S \setminus \{0\} \right\} .$$

If it possible to describe more precisely the length norm, using a famous result of Wolpert. Let  $\lambda$  be a measured lamination on the surface  $S$ . The function  $\mathbf{length}(\lambda)$  on the Teichmüller space of  $S$  is defined as follows: for each choice of a hyperbolic metric on  $S$ ,  $\mathbf{length}(\lambda)$  is the length of the corresponding measured geodesic lamination. Due to a formula of Wolpert [Wol83, Lemma 4.1], the tangent vector  $\mathbf{InfEarth}(\lambda)$  of the Teichmüller space of  $S$  at a point  $\mathbb{H}^2/\Gamma$  is the symplectic gradient of the function  $\mathbf{length}(\lambda)$  at the same point, with respect to the Weil–Petersson form of Teichmüller space:

$$d\mathbf{length}(\lambda)^\sharp = \mathbf{InfEarth}(\lambda) ,$$

in particular,

$$\|d\mathbf{length}(\lambda)\|_{\text{length}}^* = \mathbf{length}(\lambda) .$$

Theorem 3.42 together with Wolpert result gives an identification between the cotangent space of Teichmüller space and  $\mathcal{ML}_\Gamma$  [KS07, Lemma 2.3]. In consequence we obtain the following.

**Theorem 3.48** ([Thu98, Theorem 5.1]). *Let  $\alpha \in T_{\mathbb{H}^2/\Gamma}^* \text{Teich}_S$ , and  $\lambda$  such that  $\alpha = d\mathbf{length}(\lambda)$ . Then*

$$\|\alpha\|_{\text{length}}^* = \mathbf{length}(\lambda)$$

*defines an asymmetric norm on  $T_{\mathbb{H}^2/\Gamma}^* \text{Teich}_S$ .*

And finally, the length norm on  $T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$  writes as follows: for  $X \in T_{\mathbb{H}^2/\Gamma} \text{Teich}_S$ ,

$$\|X\|_{\text{length}} = \sup \left\{ \frac{d\mathbf{length}(\lambda)(X)}{\mathbf{length}(\lambda)} \mid \lambda \in \mathcal{ML}_\Gamma \setminus \{0\} \right\} . \quad (41)$$

## 4 Anosov representations

In all this section,  $\Gamma$  is a cocompact lattice of  $O_+(d, 1)$ , and  $\tau$  an element of  $Z^1(\Gamma, \mathbb{R}^{d,1})$ . We will consider the associated group  $\Gamma_\tau$  of isometries of co-Minkowski space. The aim of this section is to provide an alternative proof (Proposition 4.10) of the existence and uniqueness of the  $\tau$ -invariant map  $b_\tau$  already exhibited in Lemma 3.14. This proof involves the *Anosov character* of  $\Gamma_\tau$  as a representation of the hyperbolic group  $\Gamma$  into the group of isometries of the Minkowski space. As a by-product, we will see that the convergence in Lemma 3.27 is not only pointwise, but uniform (Lemma 4.11).

We start by the following fundamental observation: since stabilizer of points are non-compact, there is no  $O_+(d, 1) \ltimes \mathbb{R}^{d,1}$  invariant metric on the boundary of co-Minkowski space. However, if one fixes an element  $x_0$  in  $\mathcal{H}^d$ , then  $x \mapsto \langle x, x \rangle_{d,1} + 2\langle x, x_0 \rangle_{d,1}^2$  is a positive definite form, hence an Euclidean metric on  $\mathbb{R}^{d,1}$ , that we denote by  $\langle \cdot, \cdot \rangle_{x_0}$ .

The choice of  $x_0$  also induces a splitting  $\mathbb{R}^{d,1} \approx \mathbb{R}^d \times \mathbb{R}$ : here,  $\mathbb{R}$  is the linear subspace spanned by  $x_0$ , and  $\mathbb{R}^d$  is the orthogonal of  $x_0$  for the Minkowskian scalar product. Until now, when writing  $\partial^* \mathbb{R}^{d,1} \approx \partial B^d \times \mathbb{R}$ , we were always implicitly doing the choice  $x_0 = (0, \dots, 0, 1)$ , but in this section we will also consider other choices. What is relevant for us now, is that the choice of  $x_0$  induces a Riemannian metric  $g_{x_0}$  on  $\partial^* \mathbb{R}^{d,1}$ : the one making  $\partial B^d$  and  $\mathbb{R}$  orthogonal, and whose restrictions to  $\partial B^d$  and  $\mathbb{R}$  are the ones induced by the Euclidean metric  $\langle \cdot, \cdot \rangle_{x_0}$ .

Let us be more precise: we can define  $\partial^* \mathbb{R}^{d,1}$  as the space of lightlike affine hyperplanes of Minkowski space. Once fixed the unit timelike vector  $x_0$ , we can parametrize  $\partial^* \mathbb{R}^{d,1}$  by pairs  $(w, h)$  where:

- $w$  is a future lightlike vector in  $\mathbb{R}^{d,1}$  in the affine spacelike hyperplane  $H_{x_0}$  of equation  $\langle x_0, w \rangle_{d,1} = -1$  (therefore,  $H_{x_0}$  is the hyperplane tangent to  $\mathcal{H}^d$  at  $x_0$ ),
- $h$  any real number.

The associated lightlike affine hyperplane is then the one given by the equation:

$$\langle w, \cdot \rangle_{d,1} = -h .$$

The set of future lightlike vectors lying in  $H_{x_0}$  is the unit sphere in this Euclidean space. The metric  $g_{x_0}$  is the product of the usual metric on this unit sphere by the usual metric on the real line. The distance function we will actually use is not the one induced by  $g_{x_0}$  but the following one:

$$\begin{aligned} d_{x_0}((w_1, h_1), (w_2, h_2)) &= \sqrt{\langle w_1 - w_2, w_1 - w_2 \rangle_{d,1} + (h_1 - h_2)^2} \\ &= \sqrt{-2\langle w_1, w_2 \rangle_{d,1} + (h_1 - h_2)^2} . \end{aligned}$$

We will also consider the closed hyperbolic manifold  $N = \Gamma \backslash \mathcal{H}^d$ , and the geodesic flow  $\phi^t$  on the unitary tangent bundle  $M = T^1 N$ . Recall that, for any element  $v$  of  $M$ , the image  $\phi^t(v)$  is the unique vector tangent to the geodesic starting from  $v$  and at distance  $t$  along this geodesic.

Actually,  $M$  is the quotient of the unitary tangent bundle  $T^1 \mathcal{H}^d$  by the natural action of  $\Gamma$ . The unitary tangent bundle  $T^1 \mathcal{H}^d$  is also naturally identified with pairs  $(x, v)$ , where the base point  $x$  is an element of  $\mathcal{H}^d$ , and  $v$  a unit spacelike vector in Minkowski space orthogonal to  $x$ . The geodesic flow  $\tilde{\phi}^t$  on  $T^1 \mathcal{H}^d$  is then:

$$\tilde{\phi}^t(x, v) = (\cosh(t)x + \sinh(t)v, \sinh(t)x + \cosh(t)v) .$$

**Definition 4.1** (Foliated bundle over  $M$ ). *Let  $E_\tau$  be the quotient of the product  $T^1 \mathcal{H}^d \times \partial^* \mathbb{R}^{d,1}$  by the diagonal action of  $\Gamma_\tau$  —where  $\Gamma_\tau$  acts on  $\mathcal{H}^d$  through its linear part. Let  $\pi_\tau : E_\tau \rightarrow M$  be the map induced by the projection on the first factor. This map is a fibration, of fiber  $\partial^* \mathbb{R}^{d,1} / \Gamma_\tau$ . It is called the foliated bundle of holonomy group  $\Gamma_\tau$  over  $M$ .*

**Definition 4.2** (Lifted geodesic flow). Let  $\tilde{\phi}_\tau^t$  be the flow on  $T^1\mathcal{H}^d \times \partial^*\mathbb{R}^{d,1}$  defined by:

$$\tilde{\phi}^t((x, v), \xi) = (\tilde{\phi}^t(x, v), \xi) .$$

This flow commutes with the  $\Gamma_\tau$  action, and induces a flow on  $E_\tau$ , denoted by  $\phi_\tau^t$ .

We clearly have:

$$\forall t \in \mathbb{R} \quad \phi_\tau^t \circ \pi_\tau = \pi_\tau \circ \tilde{\phi}_\tau^t .$$

We also can distinguish two subbundles  $\Delta_\tau^\pm$  of  $\pi_\tau : E_\tau \rightarrow M$ . More precisely:

**Lemma 4.3.** Let  $(x, v)$  in  $T^1\mathcal{H}^d$ . Let  $(w, h)$  be an element of  $\partial^*\mathbb{R}^{d,1}$  parametrized by the pair  $(w, h)$  under the identification defined above associated to  $x$ . Then:

$$-1 \leq \langle w, v \rangle_{d,1} \leq 1 .$$

Moreover, the equality  $\langle w, v \rangle_{d,1} = 1$  holds if and only if  $w = x + v$ , and the equality  $\langle w, v \rangle_{d,1} = -1$  holds if and only if  $w = x - v$ .

*Proof.* For every  $(x, v)$  in  $T^1\mathcal{H}^d$ , and every lightlike element  $w$  of  $H_x$ ,  $v$ ,  $-v$  and  $w - x$  are unit elements in the Euclidean hyperplane  $x^\perp$ . The Lemma follows easily since  $\langle w, v \rangle_{d,1} = \langle w - x, v \rangle_{d,1}$ .  $\square$

**Definition 4.4.** We denote by  $\tilde{\Delta}^+$  (respectively  $\tilde{\Delta}^-$ ) the closed subset of  $T^1\mathcal{H}^d \times \partial^*\mathbb{R}^{d,1}$  comprising elements  $(x, v, \xi)$  such that the orthogonal of the lightlike hyperplane  $\xi$  is  $x + v$  (respectively  $x - v$ ).

The complement  $T^1\mathcal{H}^d \times \partial^*\mathbb{R}^{d,1} \setminus \tilde{\Delta}^\pm$  is an open subset that we denote by  $\tilde{\aleph}^\pm$ .

It is straightforward to check that  $\tilde{\Delta}^\pm$  and  $\tilde{\aleph}^\pm$  are  $\Gamma_\tau$ -invariant and define closed subsets  $\Delta_\tau^\pm$  and open subsets  $\aleph_\tau^\pm$  of  $E_\tau$ . Moreover:

**Lemma 4.5.**  $\tilde{\Delta}^\pm$  and  $\tilde{\aleph}^\pm$  are  $\tilde{\phi}_\tau^t$ -invariant.

*Proof.* We just have to prove that  $\tilde{\Delta}^\pm$  is  $\tilde{\phi}_\tau^t$ -invariant. We just treat the case of  $\tilde{\Delta}^+$ , the case of  $\tilde{\Delta}^-$  is similar.

Let  $(x, v, \xi)$  be an element of  $\tilde{\Delta}^+$ : it means that, for the parametrization defined by  $x$ , the lightlike hyperplane  $\xi$  is parametrized by  $(w, h)$ , where  $w = x + v$  —or, equivalently,  $\langle w, v \rangle_{d,1} = 1$  (see Lemma 4.3). Denote by  $(x_t, v_t)$  the iterate  $\tilde{\phi}_\tau^t(x, v)$ . Let  $(w_t, h_t)$  be the pair parameterizing  $\xi$  for the identification defined by  $x_t$ . Then,  $w_t = \lambda_t(x + v)$  for some positive real number  $\lambda_t$ . We must have:

$$\begin{aligned} -1 &= \langle w_t, x_t \rangle_{d,1} \\ &= \langle \lambda_t(x + v), \cosh(t)x + \sinh(t)v \rangle_{d,1} \\ &= -\lambda_t \cosh(t) + \lambda_t \sinh(t) \\ &= -\lambda_t \exp(-t) . \end{aligned}$$

Therefore  $\lambda_t = \exp(t)$ , and:

$$\begin{aligned} w_t &= \exp(t)(x + v) \\ &= (\cosh(t) + \sinh(t))x + (\cosh(t) + \sinh(t))v \\ &= (\cosh(t)x + \sinh(t)v) + (\sinh(t)x + \cosh(t)v)_{d,1} \\ &= x_t + v_t . \end{aligned}$$

The Lemma follows.  $\square$

Therefore,  $\Delta_\tau^\pm$  are  $\phi_\tau^t$ -invariant. The restriction of  $\pi_\tau$  to  $\Delta_\tau^\pm$  is a fibration, with 1-dimensional fibers. The restriction  $\pi_\tau^\pm$  to  $\aleph_\tau^\pm$  is a fibration with contractible fibers. Indeed, every fiber is the complement in  $\partial^*\mathbb{R}^{d,1}$  of a degenerate vertical line removed, i.e. the product of a 1-punctured sphere by the real line.

**Definition 4.6.** Let  $F_\tau$  be the space of continuous sections of the fibration  $\pi_\tau : E_\tau \rightarrow M$ . We equip  $F_\tau$  with the following metric:

$$D(\sigma_1, \sigma_2) = \sup_{(x,v) \in T^1\mathcal{H}^d} d_x(\sigma_1(x,v), \sigma_2(x,v)) .$$

We denote by  $F_\tau^\pm$  the open subset comprising sections of  $\pi_\tau : \mathfrak{N}_\tau^\pm \rightarrow M$ , and by  $F(\Delta^\pm)_\tau$  the space of sections of  $\pi_\tau^\pm : \Delta_\tau^\pm \rightarrow M$ .

Since  $M$  is compact, this upper bound is always attained. Observe that the metric space  $(F_\tau, D)$  is complete.

Let  $\sigma$  be an element of  $F_\tau$ . It lifts to a  $\Gamma_\tau$ -equivariant section of the fibration  $T^1\mathcal{H}^d \times \partial^*\mathbb{R}^{d,1} \rightarrow T^1\mathcal{H}^d$  and therefore provides a  $\Gamma_\tau$ -equivariant maps  $F : T^1\mathcal{H}^d \rightarrow \partial^*\mathbb{R}^{d,1}$ . Actually,  $F_\tau$  is in 1-1 correspondence with the space of  $\Gamma_\tau$ -equivariant maps from  $T^1\mathcal{H}^d$  into  $\partial^*\mathbb{R}^{d,1}$ .

The flow  $\phi_\tau$  induces a 1-parameter group of transformations on  $(F_\tau, D)$ : for every  $t$  in  $\mathbb{R}$ , and any  $\sigma$  in  $F_\tau$ , define:

$$\Phi_\tau^t(\sigma)(x,v) = \phi_\tau^t(\sigma(\phi^{-t}(x,v))) .$$

According to Lemma 4.5, the subbundles  $F_\tau^\pm$  are  $\Phi_\tau^t$ -invariant.

We can now prove the fundamental fact:

**Lemma 4.7.** *The flow  $\Phi_\tau^t$  on  $F_\tau^+$  is exponentially contracting: there exist positive real numbers  $T, a$  and  $0 < C < 1$  such that, for every  $t > T$  and for every  $\sigma_1, \sigma_2$  in  $F_\tau^+$  we have:*

$$D(\Phi_\tau^t(\sigma_1), \Phi_\tau^t(\sigma_2)) < Ce^{-at}D(\sigma_1, \sigma_2) .$$

*Proof.* Let  $F : T^1\mathcal{H}^d \rightarrow \partial^*\mathbb{R}^{d,1}$  be a  $\Gamma_\tau$ -equivariant map corresponding to elements of  $F_\tau^+$ . We denote by  $F_t$  the iterate  $\Phi_\tau^t(F)$ . Let  $(x,v)$  be an element of  $T^1\mathcal{H}^d$ . Let  $\xi$  be the image  $F(x,v)$ . It is an affine lightlike hyperplane. By definition of  $\Phi_\tau^t$ ,  $\xi$  is the image under  $F_t$  of  $\tilde{\phi}^t(x,v) = (x_t, v_t) = (\cosh(t)x + \sinh(t)v, \sinh(t)x + \cosh(t)v)$ . Let  $(w_t, h_t)$  be the pair corresponding to  $\xi$  satisfying  $\langle x_t, w_t \rangle_{d,1} = -1$  and such that  $\xi$  is the hyperplane of equation:

$$\langle w_t, \cdot \rangle_{d,1} = -h_t .$$

In particular, we see that  $-hx$  belongs to  $\xi$ , and therefore, for every  $t$  we have:

$$h_t = -h \langle w_t, x \rangle_{d,1} . \quad (42)$$

Since the lightlike vectors  $w_t$  are all orthogonal to  $\xi$ , they are proportional: for every  $t$ , there is a real number  $\lambda_t > 0$  such that  $w_t = \lambda_t w_0$ . From equation (42) we see:

$$h_t = h \lambda_t .$$

A straightforward computation shows:

$$\lambda_t = \frac{1}{\cosh(t) - \sinh(t) \langle v, w_0 \rangle_{d,1}} .$$

Let now  $F_1, F_2$  be two  $\Gamma_\tau$ -equivariant maps from  $T^1\mathcal{H}^d$  into  $\partial^*\mathbb{R}^{d,1}$  corresponding to sections of  $\pi_\tau : \mathfrak{N}_\tau^+ \rightarrow M$ . The distance in  $F_\tau$  between the corresponding sections is then the supremum of  $d_x(F_1(x,v), F_2(x,v))$  where  $(x,v)$  describes  $T^1\mathcal{H}^d$ . Applying  $\Phi_\tau^t$  simply means that we replace  $F_1$  and  $F_2$  by  $F_1 \circ \tilde{\phi}_\tau^{-t}$  and  $F_2 \circ \tilde{\phi}_\tau^{-t}$ . It follows that the distance after applying  $\Phi_\tau^t$  is the supremum of  $d_{x_t}(F_1(x,v), F_2(x,v))$  where  $(x,v)$  describes  $T^1\mathcal{H}^d$  and where  $x_t$  denotes as above the  $x$  component of  $\tilde{\phi}^t(x,v)$ , i.e.  $\cosh(t)x + \sinh(t)v$ .

The computation above shows that, for  $i = 1, 2$ , the pair  $(w_t^i, h_t^i)$  representing  $F_i(x,v)$  satisfy:

$$\begin{aligned} w_t^i &= \frac{w_0^i}{\cosh(t) - \sinh(t) \langle v, w_0 \rangle_{d,1}} , \\ h_t^i &= \frac{h_0^i}{\cosh(t) - \sinh(t) \langle v, w_0 \rangle_{d,1}} . \end{aligned}$$

Therefore,  $d_{x_t}(F_1(x, v), F_2(x, v)) = \frac{d_{x_0}(F_1(x, v), F_2(x, v))}{\cosh(t) - \sinh(t)\langle v, w_0 \rangle_{d,1}}$ . Since the  $F_1$  and  $F_2$  correspond to sections in  $F_\tau^+$ , we have:

$$-1 \leq \langle v, w_0 \rangle_{d,1} < 1 .$$

It follows that for big  $t$ , the quantity  $\cosh(t) - \sinh(t)\langle v, w_0 \rangle_{d,1}$  is equivalent to  $e^t(1 - \langle v, w_0 \rangle_{d,1})/2$ . The Lemma follows.  $\square$

**Corollary 4.8.** *There exists one and only one  $\Phi_\tau^t$ -invariant section  $\sigma_\tau^+$  of  $\pi_\tau : \mathbb{N}_\tau^+ \rightarrow M$ . This invariant section actually takes value in  $\Delta_\tau^-$ .*

*Proof.* Let  $T > 0$  be a real number big enough so that  $\Phi_\tau^T$  is contracting. Since  $\Delta_\tau^-$  is a subbundle of  $\mathbb{N}_\tau^+$ ,  $F(\Delta^-)_\tau$  is a closed subset of  $F_\tau^+$ . The restriction  $D$  to  $F(\Delta^-)_\tau$  is therefore complete. Hence, as any contracting map acting on a complete metric space,  $\Phi_\tau^T$  admits a unique fixed point  $\sigma_\tau^+$  in  $F(\Delta^-)_\tau$ . Since its action on  $F_\tau^+$  is contracting too,  $\sigma_\tau^+$  is the unique fixed point in  $F_\tau^+$ . Since  $\Phi_\tau^T$  commutes with  $\Phi_\tau^t$  for every real number  $t$ ,  $\sigma_\tau^+$  is fixed by every  $\Phi_\tau^t$ .  $\square$

Let  $F_\tau : T^1\mathcal{H}^d \rightarrow \partial^*\mathbb{R}^{d,1}$  be the  $\Gamma_\tau$ -equivariant lifting of the  $\Phi_\tau^t$ -invariant section  $\sigma_\tau^+$  exhibited in Corollary 4.8. The  $\Phi_\tau^t$ -invariance means that  $F_\tau$  is constant along the orbits of the geodesic flow  $\tilde{\phi}^t$  of  $T^1\mathcal{H}^d$ . The following Lemma shows that we have much more:

**Lemma 4.9.** *The map  $F_\tau$  is constant along the leaves of the weak unstable foliation of the geodesic flow  $\phi^t$ .*

*Proof.* Let  $\theta_1, \theta_2$  be two orbits of  $\tilde{\phi}^t$  in the same *unstable leaf*, i.e. such that for every  $(x_1, v_1)$  in  $\theta_1$  and every  $(x_2, v_2)$  in  $\theta_2$  the isotropic vectors  $x_1 - v_1$  and  $x_2 - v_2$  are proportional, i.e. represent the same element of  $\partial\mathcal{H}^d$ . On the other hand, since the invariant section takes value in  $\Delta_\tau^-$ , we have that  $F_\tau(x_1, v_1)$  and  $F_\tau(x_2, v_2)$  are lightlike hyperplanes orthogonal to respectively  $x_1 - v_1$  and  $x_2 - v_2$ . Therefore, they are parallel.

Let  $p_1, p_2$  be the projections of  $(x_1, v_1)$  and  $(x_2, v_2)$  in  $M$ . Then, by replacing  $p_2$  by another element of its  $\phi^t$ -orbit, one can assume that  $p_1$  and  $p_2$  lies in the same *strong unstable leaf*, i.e. that the hyperbolic distance between  $\phi^t(p_1)$  and  $\phi^t(p_2)$  converge exponentially to 0 when  $t$  goes to  $-\infty$ .

It follows that the hyperbolic distance between  $\tilde{\phi}^t(x_1, v_1)$  and  $\tilde{\phi}^t(x_2, v_2)$  converges to 0 when  $t$  is going to  $-\infty$ . Let  $\xi_1 = F_\tau(x_1, v_1)$  and  $\xi_2 = F_\tau(x_2, v_2)$ . Since  $F_\tau$  is (uniformly) continuous, it follows that  $d_t(\xi_1, \xi_2)$  converges to 0, where  $d_t$  is the distance on  $\partial^*\mathbb{R}^{d,1}$  defined by  $\tilde{\phi}^t(x_1, v_1)$ . But this is almost a contradiction with Lemma 4.7, that shows that this distance should be exponentially increasing when  $t$  is going to  $-\infty$ . The only possibility is that this distance is actually vanishing, i.e.  $\xi_1 = \xi_2$ . The Lemma is proved.  $\square$

In the sequel, we use the cylindrical affine model of the co-Minkowski space, i.e. write elements of  $\partial^*\mathbb{R}^{d,1}$  as pairs  $(w, h)$  where  $w$  is a lightlike vector satisfying  $\langle x_0, w \rangle_{d,1} = -1$ , where  $x_0$  denotes the element  $(0, \dots, 0, 1)$  of  $\mathbb{R}^{d,1}$ .

**Proposition 4.10.** *There is a continuous map  $b_\tau : \partial B^d \rightarrow \mathbb{R}$  such that the  $\Gamma_\tau$ -equivariant map  $F_\tau : T^1\mathcal{H}^d \rightarrow \partial^*\mathbb{R}^{d,1}$  is given by:*

$$(x, v) \mapsto \left( -\frac{x-v}{\langle x_0, x-v \rangle_{d,1}}, b_\tau \left( -\frac{x-v}{\langle x_0, x-v \rangle_{d,1}} \right) \right) .$$

*Proof.* We still parameterize the unit tangent bundle of the hyperbolic space by pairs  $(x, v)$  where  $x$  is a unit timelike vector and  $v$  a unit spacelike vector orthogonal to  $x$ .

Since the invariant section takes value in the subbundle  $\Delta^-$ , the map  $F_\tau$  is such that  $F_\tau(x, v) = (w(x, v), h(x, v))$  where  $w(x, v)$  is proportional to  $x - v$ , hence is equal to  $-\frac{x-v}{\langle x_0, x-v \rangle_{d,1}}$ . Moreover, according to Lemma 4.9,  $h(x, v)$  depends only on  $x - v$ , hence, only on  $-\frac{x-v}{\langle x_0, x-v \rangle_{d,1}}$ . Therefore,  $F_\tau$  is given by:

$$(x, v) \mapsto \left( -\frac{x-v}{\langle x_0, x-v \rangle_{d,1}}, b_\tau \left( -\frac{x-v}{\langle x_0, x-v \rangle_{d,1}} \right) \right)$$

for some map  $b_\tau : \partial B^d \rightarrow \mathbb{R}$ . □

As a corollary, we get the following amelioration of Lemma 3.27:

**Lemma 4.11.** *Let  $\tau_n \rightarrow \tau$ . Then  $b_{\tau_n}$  (resp.  $h_{\tau_n}^\pm, h_{\tau_n}^{\text{mean}}$ ) converge uniformly to  $b_\tau$  (resp.  $h_\tau^\pm, h_\tau^{\text{mean}}$ ).*

*Proof.* We just give a sketch of proof. First, we observe that we just have to prove the statement for  $b_{\tau_n}$ , since the uniform convergence of  $h_{\tau_n}^\pm$  (resp.  $h_{\tau_n}^{\text{mean}}$ ) follows then from Lemma 2.43 (resp. Lemma 2.38). The key point is that when  $n$  is big enough, the fibration  $\pi_{\tau_n} : E_{\tau_n} \rightarrow M$  is isomorphic to the fibration  $\pi_\tau : E_\tau \rightarrow M$ . More precisely, (the inverse of) this isomorphism of fibrations send the graph of the section  $\sigma_\tau$  to the graph of some section which is already a nearly fixed point for  $\Phi_{\tau_n}^t$ . The bigger  $n$  is, the closer (for the metric  $D$ ) is this nearly fixed point to the eventual fixed point  $\sigma_{\tau_n}^\pm$ . In other words, the bigger is  $n$ , the closer to  $\sigma_\tau^\pm$  is  $\sigma_{\tau_n}^\pm$  for the compact-open topology. The Lemma clearly follows, due to the form of the lifts  $F_\tau$  and  $F_{\tau_n}$  given by Proposition 4.10. □

**Remark 4.12.** *Mutandi mutandis*, one can show that there is also a unique fixed point for  $\Phi_\tau^t$  in  $F_\tau^-$ , which is this time an exponential repeller, and which is actually a section of the subbundle  $\Delta^+$ . It provides, as in Proposition 4.10 a map from  $\partial B^d$  into  $\mathbb{R}$ , which is actually the map  $b_\tau$ . Details are left to the reader.

**Remark 4.13.** Instead of considering the fiber bundle  $\mathbb{N}_\tau^\pm$ , one might have restricted the study to the subbundles  $\Delta^\pm$ , which are simpler since with one-dimensional fibers. However, the most efficient way to deal with these bundles is to consider them as subbundles of  $E_\tau$ .

We conclude this section by an interpretation of its content in term of Anosov representations. Let  $G$  be a general Lie group acting on some space  $X$ , and let  $\rho : \Gamma \rightarrow G$  be a representation. Consider as in Definition 4.1 the foliated bundle  $\pi_\rho : E_\rho(X) \rightarrow M$  where  $E_\rho(X)$  is the quotient of the product  $T^1\mathcal{H}^d \times X$  by the diagonal action of  $\Gamma$  —where the action of  $\Gamma$  on  $X$  is given by  $\rho$ . As in Definition 4.2, the geodesic flow  $\phi^t$  lifts to some horizontal flow  $\phi_\rho^t$  on  $E_\rho(X)$  so that the bundle map  $\pi_\rho$  is equivariant.

The representation  $\rho$  is said  $(G, X)$ -Anosov if the following holds: there is a section  $\sigma : M \rightarrow E_\rho(X)$  which is equivariant for the flows, and such that the graph  $\Lambda$  of  $\sigma$  is a *closed hyperbolic subset* for the lifted flow  $\phi_\rho^t$ : it means that the restriction  $T_\Lambda E_\rho(X)$  of the tangent bundle of  $E_\rho(X)$  to  $\Lambda$  splits as a Whitney sum of subbundles  $E^+ \oplus E^- \oplus \Phi$ , where:

- $\Phi$  is the one dimensional bundle tangent to the flow  $\phi_\rho^t$ ,
- $E^+$  is exponentially contracted by the flow,
- $E^-$  is exponentially expanded by the flow.

For more details, see [Lab06] or [Bar10, Bar15].

In our case, the inclusion  $\Gamma \approx \Gamma_\tau \subset SO_+(d, 1) \times \mathbb{R}^{d,1}$  is  $(G, X)$ -Anosov where  $X$  is the space of oriented  $(d - 1)$ -dimensional spacelike affine subspaces of  $\mathbb{R}^{d,1}$ . Indeed,  $X$  identifies with the open domain in  $\partial^*\mathbb{R}^{d,1} \times \partial^*\mathbb{R}^{d,1}$  made of pairs  $(\xi_1, \xi_2)$ , where  $\xi_1$  and  $\xi_2$  are non parallel affine lightlike hyperplanes. Therefore, the two equivariant sections  $\sigma_\tau^\pm$  define altogether a section  $\sigma$  of  $\pi_\rho : E_\rho(X) \rightarrow M$ . Moreover, it follows from Lemma 4.7 and Remark 4.12 that the graph of  $\sigma$  is a closed hyperbolic subset for  $\phi_\rho^t$ .

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