Hyperbolic geometry of shapes of convex bodies

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Abstract

We define a distance on the space of convex bodies in the n-dimensional Euclidean space, up to translations and homotheties, which makes it isometric to a convex subset of the infinite dimensional hyperbolic space. The ambient Lorentzian structure is an extension of the intrinsic area form of convex bodies.

We deduce that the space of shapes of convex bodies (i.e. convex bodies up to similarities) has a proper distance with curvature bounded from below by -1. In dimension 3, this space naturally identifies with the space of distances with non-negative curvature on the 2-sphere.

Contents

1 Introduction

Intrinsic area of convex bodies

A *convex body* is a non-empty compact convex subset of $\mathbb{R}^n$. In this article, we set $n > 1$. For a plane convex body K (i.e. a convex body in $\mathbb{R}^2$), speaking about the "area" of K usually means to look at its two dimensional Lebesgue measure. Note that the area of plane convex bodies is positively homogeneous of degree 2: for $\lambda > 0$, $\mathbf{area}(\lambda K) = \lambda^2 \mathbf{area}(K)$. For a convex body in $\mathbb{R}^3$, the "area" usually refers to its surface area, i.e. the 2-dimensional total Hausdorff measure of its boundary ∂K. Here also, the surface area is positively homogeneous of degree two.

There are two ways to generalize the notion of "area" to convex bodies in $\mathbb{R}^n$ for $n > 3$. Both are coming from the Steiner Formula. Let B^n be the closed unit ball centred at the origin in $\mathbb{R}^n$, and let κ_n be its volume. Let us set $\kappa_0 = 1$ and $\kappa_1 = 2$. If K is a convex body in $\mathbb{R}^n$, then there exists non-negative real numbers $V_i(K)$, $i = 0, \dots, n$ such that, for any $\epsilon > 0$,

$$\mathrm{vol}_n(K + \epsilon B^n) = \sum_{i=0}^{n} \epsilon^{n-i} \kappa_{n-i} V_i(K) \ . \tag{1.1}$$

Here vol_n is the Lebesgue measure of $\mathbb{R}^n$, and the sum is the Minkowski addition: $A + B = \{a + b | a \in A, b \in B\}$. It appears that $V_0(K) = 1$ and $V_n(K) = \mathrm{vol}_n(K)$.

The first way to generalize the notion of surface area of convex bodies in $\mathbb{R}^3$ is to consider $V_{n-1}(K)$ as the "area", given by the first order variation of $\mathrm{vol}_n(K + \epsilon B^n)$, seen as a function of ϵ. Note that this "area" is homogeneous of degree $(n-1)$, and that for $n = 2$, this is related to the perimeter of the convex body and not to its area.

Instead, in this paper, we will consider the *intrinsic area* $V_2(K)$. Let us mention some relevant properties.

A1) For any $\lambda > 0$, $V_2(\lambda K) = \lambda^2 V_2(K)$;

A2) $V_2(K) \geq 0$;

A3) $K_1 \subset K_2 \Rightarrow V_2(K_1) \leq V_2(K_2)$;

A4) $V_2(K) = 0$ if and only if K is a point or a segment;

A5) for any $A \in \mathrm{SL}(n, \mathbb{R})$ and $p \in \mathbb{R}^n$, $V_2(A(K) + \{p\}) = V_2(K)$;

A6) Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a linear isometric embedding. Then $V_2(\iota(K)) = V_2(K)$.

These properties are well-known, however we will give proofs. Note that Property A1) above is straightforward from (1.1), as well as A5), by invariance of the Lebesgue measure. It is also clear from (1.1) that points and segments have zero intrinsic area, see Figure 1. The property A6) explains the denomination "intrinsic". For example, if $n = 2$, then $V_2(K)$ is the $\mathbb{R}^2$ Lebesgue measure of K. If $n = 3$, then the intrinsic area is half of the surface area (see [32]). If K is a convex body in $\mathbb{R}^3$ contained in a 2-plane P, then the surface area of K is two times the area of K in P: the area has to be taken into account two times, as K has two support planes that coincide with P. This is coherent with property A6), that says that the intrinsic area of K does not depend on the dimension of the ambient space.

For proofs and comments around property A6), see see pp. 208 and 214 in [32], as well as [27, 28] and Proposition 3.2 in [13] and the references given there.

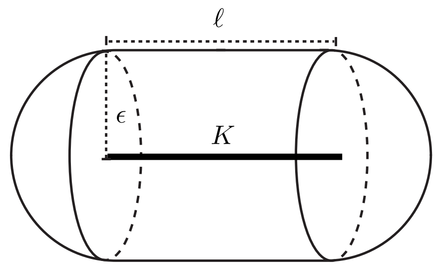

Figure 1: Let K be a segment of length ℓ in $\mathbb{R}^n$. Then $\mathrm{vol}_n(K+\epsilon B^n)$ is the sum of the volume of two half n-dimensional balls of radius ϵ, plus ℓ times the volume of a $(n-1)$-dimensional ball. By (1.1), $V_2(K)=0$ and $V_1(K)=\ell$.

There is a geometric interpretation of the intrinsic area of a convex body K that we won't use: it is, up to a dimensional constant, the mean value of the areas of the orthogonal projections of K onto two dimensional vector planes [32, (5.72)].

In the sequel we will also consider $V_1(K)$, which is related to the *mean width* of the convex body K, see Remark 4.12. We will see in Section 2.4 that this quantity is also intrinsic. Let us note that V_1 of a segment is its length, see Figure 1.

For future references, let us give the following example. Writing $\mathrm{vol}_n(B^n+\epsilon B^n) = \mathrm{vol}((1+\epsilon)B^n) = (1+\epsilon)^n\kappa_n$, from (1.1) we have $\kappa_n \sum_{i=0}^n \binom{n}{i}\epsilon^{n-i} = \sum_{i=0}^n \epsilon^{n-i}\kappa_{n-i}V_i(B^n)$ so, using classical equalities, we obtain,

$$V_2(B^n) = (n-1)\frac{n\kappa_n}{2\kappa_{n-2}} = (n-1)\pi \ , \tag{1.2}$$

and also, if $W_n = \int_0^{\pi/2}\cos^n$ is the Wallis' integral,

$$V_1(B^n) = \frac{n\kappa_n}{\kappa_{n-1}} = \frac{\mathrm{vol}(\mathbb{S}^{n-1})}{\kappa_{n-1}} = 2\pi\frac{\kappa_{n-2}}{\kappa_{n-1}} = \frac{\pi}{W_{n-1}} = 2nW_n \sim 2n\sqrt{\frac{\pi}{2n}} = \sqrt{2\pi n} \ . \tag{1.3}$$

Note that in particular we have $V_1(B^1)=2$. Let us introduce the following dimensional constants:

$$r_1(n) = V_1(B^n)^{-1},\ r_2(n) = V_2(B^n)^{-1/2} \ , \tag{1.4}$$

which are such that a n-th dimensional ball of radius $r_1(n)$ has $V_1=1$, and a n-th dimensional ball of radius $r_2(n)$ has $V_2=1$.

Mixed-area of convex bodies

The (intrinsic) area can be "polarized", in the sense that there exists a function called the (intrinsic) *mixed-area* $V_2(\cdot,\cdot)$, that can be defined as

$$V_2(K_1,K_2) = \frac{1}{2}\left(V_2(K_1+K_2) - V_2(K_1) - V_2(K_2)\right) \ , \tag{1.5}$$

and satisfying the following properties:

M1) $V_2(K_1,K_1) = V_2(K_1)$;

M2) $V_2(K_1,K_2) = V_2(K_2,K_1)$;

M3) $V_2(K_1+K_2,K_3) = V_2(K_1,K_3) + V_2(K_2,K_3)$;

M4) for $\lambda > 0$, $V_2(\lambda K_1, K_2) = \lambda V_2(K_1, K_2)$;

M5) $K_1 \subset K_2 \Rightarrow V_2(K_1, K_3) \leq V_2(K_2, K_3)$;

M6) K is a point if and only if for any convex body Q, $V_2(K, Q) = 0$;

M7) $V_2(K_1, K_2) \geq 0$; and $V_2(K_1, K_2) = 0$ if and only if K_1 or K_2 is a point, or both are segments with the same direction;

M8) we have

$$\delta(K_1, K_2) = V_2(K_1, K_2)^2 - V_2(K_1)V_2(K_2) \geq 0 \tag{1.6}$$

and if K_1 and K_2 are not points, then equality occurs if and only if K_1 and K_2 differ by a translation and a positive homothety;

M9) we have

$$V_2(K_1, K_2) \geq \sqrt{V_2(K_1)V_2(K_2)}$$

and if K_1 and K_2 are not points, then equality occurs if and only if K_1 and K_2 differ by a translation and a positive homothety.

Only properties M1) and M2) are obvious from (1.5). The property M8) is a particular case of the famous Alexandrov–Fenchel inequality. M9) is M8) written with the help of M7).

For future references, let us note the following particular case. Developing using Steiner formula (1.1) both sides of $\mathrm{vol}_n((K + B^n) + \epsilon B^n) = \mathrm{vol}_n(K + (1 + \epsilon)B^n)$ and using (1.2) and (1.3), we have $V_2(K + B^n) = V_2(K) + V_1(B^{n-1})V_1(K) + V_2(B^n)$, and from (1.5) we finally obtain

$$V_2(K, B^n) = \frac{1}{2}V_1(B^{n-1})V_1(K)\ . \tag{1.7}$$

If $K \subset \mathbb{R}^2$, then $V_1(K)$ is half the perimeter $\mathrm{per}(K)$, and $V_2(K)$ is the area $\mathrm{vol}_2(K)$, so property M8) is the isoperimetric inequality: indeed it reads

$$\begin{aligned} 0 \leq V_2(K, B^2)^2 - V_2(K)V_2(B^2) &= \frac{1}{4}V_1(B^1)^2V_1(K)^2 - \mathrm{vol}_2(K)V_2(B^2) \\ &= \frac{\mathrm{per}(K)^2}{4} - \pi\,\mathrm{vol}_2(K)\ . \end{aligned}$$

Even if the space of convex bodies is not a vector space, from its properties the mixed-area reminds a symmetric bilinear form, whose kernel is the space of points, and whose isotropic cone is the space of points and segments. Moreover the Alexandrov–Fenchel inequality (1.6) reminds a reversed Cauchy–Schwarz inequality. In the present paper we give the good framework to formalize those analogies. Apart from our main results that are stated below, we will also give proofs of the properties mentioned above within this framework —although they are classical, as intrinsic volumes are particular cases of mixed-volumes [32].

The area metric on the space of shapes

In the sequel we denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$, and by $\mathcal{K}^{n*}$ the subset of convex bodies of positive intrinsic area. In other terms, by A2) and A4), $\mathcal{K}^{n*}$ is $\mathcal{K}^n$ minus points and segments.

By property M9) of the mixed-area, for any $K_1, K_2 \in \mathcal{K}^{n*}$ we can set

$$\tilde{d}_1(K_1, K_2) = \mathrm{argch}\left(\frac{V_2(K_1, K_2)}{\sqrt{V_2(K_1)V_2(K_2)}}\right)\ .$$

This is clear that this quantity is invariant under positive homotheties of K_1 and K_2. Moreover, by A5) and (1.5), for all $p \in \mathbb{R}^n$,

$$V_2(K_1 + \{p\}, K_2) = V_2(K_1, K_2 + \{p\}) = V_2(K_1, K_2) \ ,$$

hence $\tilde{d}_1$ is invariant under translations of K_1 or K_2. By the case of equality in property M9), $\tilde{d}_1(K_1, K_2) = 0$ if and only if K_1 is the image of K_2 by a translation and a positive homothety.

Let us define the space $\mathscr{O}\mathscr{S}hape^n$ (resp. $\mathscr{O}\mathscr{S}hape^{n*}$) as the quotient of $\mathcal{K}^n$ (resp. $\mathcal{K}^{n*}$) by the action of translations and positive homotheties. For a convex body K, we denote by $[K]$ the *oriented shape* of K, that is the equivalence class of K for the action of translations and positive homotheties. We can then define the *area metric* on $\mathscr{O}\mathscr{S}hape^{n*}$: for any $[K_1], [K_2] \in \mathscr{O}\mathscr{S}hape^{n*}$ we set

$$d_1([K_1], [K_2]) = \tilde{d}_1(K_1, K_2) \ .$$

Let $K_1, K_2 \in \mathcal{K}^{n*}$. Assume that $V_2(K_1) = V_2(K_2) = a > 0$. Assume also that $[K_1] \neq [K_2]$, that is, K_1 is not the image of K_2 by a translation and a positive homothety. Consider the following equation:

$$V_2((1-t)K_1 + tK_2) = 0 \ . \tag{1.8}$$

By properties of the mixed-area, the left-hand side is a polynomial in t, and the coefficient of t^2 is $2a - 2V_2(K_1, K_2)$. Since $[K_1] \neq [K_2]$, by Alexandrov-Fenchel's inequality M8) we have $V_2(K_1, K_2) > a$: the coefficient of t^2 is negative, in particular this is a second order polynomial. An easy calculation shows that its discriminant is equal to $4\delta(K_1, K_2) > 0$ (see (1.6)). Let $t_1 < 0 < 1 < t_2$ be the two solutions of the equation (1.8), and let us define

$$\tilde{d}_2(K_1, K_2) = \frac{1}{2} \ln[0, 1, t_1, t_2] \ ,$$

where $[0, 1, t_1, t_2]$ is the cross-ratio (see Section 3.2). By (1.8), it is clear that $\tilde{d}_2$ is invariant by translation of K_1 or K_2. Let $[K_1], [K_2] \in \mathscr{O}\mathscr{S}hape^{n*}$, and let K_1, K_2 be two representatives having the same intrinsic area. We can then define

$$d_2([K_1], [K_2]) = \tilde{d}_2(K_1, K_2) \ ,$$

if $[K_1] \neq [K_2]$, and zero otherwise.

Classical trigonometry will show that $d_1 = d_2$ (see Section 3.2). We will denote it as $d_{\mathscr{O}\mathscr{S}^n}$. For futur reference, let us note that in particular we have, by (1.3) and (1.7),

$$d_{\mathscr{O}\mathscr{S}^n}([B^n], [K]) = \operatorname{argch}\left(\sqrt{\frac{n-1}{\pi}} W_{n-1} \frac{V_1(K)}{\sqrt{V_2(K)}} \right) \ . \tag{1.9}$$

The main part of the present article is to prove the following properties for $d_{\mathscr{O}\mathscr{S}^n}$.

Theorem 1. *($\mathscr{O}\mathscr{S}\mathrm{hape}^{n*}, d_{\mathscr{O}\mathscr{S}^n}$) is a uniquely geodesic proper metric space of infinite Hausdorff dimension and infinite diameter. The unique shortest path between $[K_1]$ and $[K_2]$ is $[(1-t)K_1 + tK_2]$, $t \in [0,1]$. Any element of $\mathscr{O}\mathscr{S}\mathrm{hape}^{n*}$ is the endpoint of a shortest path that is not extendable beyond this point. Moreover, ($\mathscr{O}\mathscr{S}\mathrm{hape}^{n*}, d_{\mathscr{O}\mathscr{S}^n}$) has curvature bounded from below and above by -1 in the sense of Alexandrov.*

Its boundary is $\mathscr{O}\mathscr{S}\mathrm{hape}^n \setminus \mathscr{O}\mathscr{S}\mathrm{hape}^{n*}$ and is homeomorphic to the real projective space of dimension $(n-1)$.

Let us recall the following definitions and basic properties:

- a metric space is *geodesic* if any two points are joined by a shortest path, it is *uniquely geodesic* if the shortest path is unique;

- a metric space is *proper* if every bounded closed subset is compact, and a proper metric space is locally compact and complete;
- a shortest path is *extendable* if it is strictly contained in another shortest path;
- the boundary of a metric space is the set of equivalence classes of geodesic rays at bounded distance, endowed with a natural topology [9].
- see Definition 5.1 for the property to have bounded curvature in the sense of Alexandrov.

Some assertions in Theorem 1 are rather straightforward. The description of the boundary follows because it is the set of segments up to translations and positive homotheties. $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ is infinite dimensional because it contains finite dimensional hyperbolic convex polyhedra of arbitrary dimension constructed in [3] and [17].

The fact that $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ is proper is non immediate, as it will follow from (the Blaschke selection theorem and) a theorem of R.A. Vitale, see Section 4.3.

The other properties stated in Theorem 1 will be a consequence of the description of the extrinsic geometry of $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$: it is isometric to a convex subset of a hyperbolic space of infinite dimension, see the next section.

It is interesting to compare the fact that $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ has curvature bounded from above and below to the following classical result of V. Beretovskij: any locally compact metric space with curvature bounded from above and below and such that shortest paths are extendable is isometric to a finite dimensional Riemannian manifold [6, 5]. Here we have a strong property of non-extendability, proved in Section 4.4.

Convex subset of infinite dimensional hyperbolic space

Mimicking the finite dimensional case, given any separable Hilbert space H of infinite dimension, one can define a subset which will deserve the name infinite dimensional hyperbolic space, see Section 3. In the present paper, we will define the infinite dimensional hyperbolic space $\mathbb{H}^\infty_n$ by using as Hilbert space a vector subspace of the Sobolev space $H^1(\mathbb{S}^{n-1})$, endowed with a bilinear form related to the intrinsic area, see Section 2.1. The support function of convex bodies will give a map from $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ into $\mathbb{H}^\infty_n$. More precisely we have the following.

Theorem 2. $(\mathscr{O}\mathscr{S}\mathrm{hape}^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ *is isometric to an infinite dimensional closed convex subset with empty interior and ideal points of* $\mathbb{H}^\infty_n$*.*

The construction of the infinite dimensional hyperbolic space $\mathbb{H}^\infty_n$ actually depends on n. Although for different n, all the resulting $\mathbb{H}^\infty_n$ are isometric (as separable infinite dimensional Hilbert spaces are all isometric), it is interesting to keep in mind this dependence, as we will define canonical totally geodesic isometric embeddings of $\mathbb{H}^\infty_n$ into $\mathbb{H}^\infty_{n+1}$, that have the following interpretation for convex bodies. For P a k-dimensional vector subspace of $\mathbb{R}^n$, let us denote by $\mathscr{O}\mathscr{S}hape^{n*}_P$ the subset of $\mathscr{O}\mathscr{S}hape^{n*}$ made of oriented shapes of convex bodies in $\mathbb{R}^n$ contained in an affine space directed by P. It will be obvious from the proof of Theorem 2 that $\mathscr{O}\mathscr{S}hape^{n*}_P$ is a convex subspace of $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$, isometric to $(\mathscr{O}\mathscr{S}hape^{k*}, d_{\mathscr{O}\mathscr{S}^k})$. See also Section 6.

We don't know if some results of hyperbolic geometry can be translated into results for convex bodies.

Space of shapes

In Section 5 we investigate $\mathscr{S}hape^{n*}$, the quotient of $\mathscr{O}\mathscr{S}hape^{n*}$ by linear isometries of the Euclidean space $\mathbb{R}^n$: $\mathscr{S}hape^{n*}$ is the space of convex bodies in $\mathbb{R}^n$ (not reduced to points or segments) up to Euclidean similarities.

From general properties of quotient of metric spaces with curvature bounded from below by -1 (in short, CBB(-1)) by a compact isometry group (see Section 5), the metric $d_{\mathscr{O}\mathscr{S}^n}$ induces a metric $d_{\mathscr{S}^n}$ on $\mathscr{S}hape^{n*}$, with the following properties.

Theorem 3. *($\mathscr{S}$hape$^{n*}, d_{\mathscr{S}^n}$) is a CBB(-1) proper geodesic metric space with boundary reduced to a point. It is not uniquely geodesic. It contains many totally geodesic hyperbolic surfaces.*

The fact that ($\mathscr{S}hape^{n*}, d_{\mathscr{S}^n}$) is not uniquely geodesic implies that it is not CAT(0), hence not CAT(-1). However we don't know if it could be locally CAT(0).

The case $n = 3$ is of some interest, as by famous results of Alexandrov and Pogorelov, there is a natural homeomorphism between ($\mathscr{S}hape^{3*}, d_{\mathscr{S}^3}$) and the space of metrics of non-negative curvature on the sphere $\mathbb{S}^2$ of total area one, up to isometries (endowed with the Gromov–Hausdorff topology). In turn, we are providing a CBB(-1) metric on this space of metrics on the sphere, see Section 5.7. This is close from a celebrated construction of W.P. Thurston [34] using the area form in order to give a (complex) hyperbolic structure on the space of flat metrics with prescribed conical singularities of positive curvature on the sphere $\mathbb{S}^2$, up to orientation-preserving isometries —see [16] for the relations between Thurston's construction and the one of the present paper.

As an example of open questions related to the present work, the last section of the present paper introduce the inductive limits of $\mathscr{O}\mathscr{S}hape^{n*}$ and $\mathscr{S}hape^{n*}$, that is allowed by the intrinsic nature of the distances we defined. They are spaces of the oriented shapes and shapes of all the convex bodies.

Let us mention the fact that a similar construction can be performed for spaces of convex set in a Lorentzian Minkowski space (instead of convex sets of the Euclidean space as in the present paper), which are invariant under a cocompact lattice of linear isometries. For such sets, a Steiner formula holds, but the intrinsic area form leads to a positive definite form on the suitable Sobolev space. In turn, one obtains a convex subset of an infinite dimensional spherical (instead of hyperbolic) space. We refer to [15, 16] for more details.

Notations

To avoid confusions, let us describe here the notations we will use in the article. Everything that appears here will be defined precisely later.

$\mathcal{K}^n$ (resp. $\mathcal{K}^{n*}$) is the set of convex bodies in $\mathbb{R}^n$ (resp. convex bodies with positive intrinsic area), and $\mathcal{K}^n_S$ (resp. $\mathcal{K}^{n*}_S$) is the subset of convex bodies with Steiner point at the origin (resp. convex bodies with positive intrinsic area).

In the sequel, a star as upperscript mean that we consider only convex bodies with positive intrinsic area (that is, we exclude points and segments). In the following table, it is obvious that all the sets without a star are in bijection, as well as all the sets with a star:

convex bodies in $\mathbb{R}^n$...	**up to positive homotheties**	**with $V_2 = 1$**	**with $V_1 = 1$**
up to translations	$\mathscr{O}\mathscr{S}hape^n$ and $\mathscr{O}\mathscr{S}hape^{n*}$		
with Steiner point at the origin	$\mathcal{K}^n_{SH}$ and $\mathcal{K}^{n*}_{SH}$	$\mathcal{K}^n_{SV_2}$ and $\mathcal{K}^{n*}_{SV_2}$	$\mathcal{K}^n_{SV_1}$ and $\mathcal{K}^{n*}_{SV_1}$

We have $\mathrm{Supp}(\mathcal{K}^{n*}_S) \subset \mathcal{C}_n$ and

$$\mathrm{Supp}(\mathcal{K}^{n*}_{SH}) \subset \mathbb{H}^\infty_n, \quad \mathrm{Supp}(\mathcal{K}^{n*}_{SV_2}) \subset \mathcal{H}^\infty_n, \quad \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}) \subset \mathbb{K}\mathbf{lein}^\infty_n \ .$$

The map Supp defines isometries

$$\begin{aligned}
(\mathcal{K}^{n*}_{SH}, d_{SH}) &\overset{\sim}{\longrightarrow} (\mathrm{Supp}(\mathcal{K}^{n*}_{SH}), d_{\mathbb{H}}) \ ,\\
(\mathcal{K}^{n*}_{SV_2}, d_{SV_2}) &\overset{\sim}{\longrightarrow} (\mathrm{Supp}(\mathcal{K}^{n*}_{SV_2}), d_{\mathcal{H}}) \ ,\\
(\mathcal{K}^{n*}_{SV_1}, d_{SV_1}) &\overset{\sim}{\longrightarrow} (\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}}) \ ,
\end{aligned}$$

and all these sets are isometric to $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$.

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2 Infinite dimensional Minkowski space

In this section, we recall elementary properties of the spherical Laplacian and we introduce the bilinear form $\overline{V_2}^n$ on a Sobolev space, that will be the main object of this paper.

2.1 Laplacian on the sphere

Let us denote by $\|\cdot\|_{L^2}$ the L^2 norm on the round sphere $\mathbb{S}^{n-1}$. Let $H^1(\mathbb{S}^{n-1})$ be the Sobolev space of $\mathbb{S}^{n-1}$, i.e. the space of functions $\mathbb{S}^{n-1}\to\mathbb{R}$ which are in $L^2(\mathbb{S}^{n-1})$ as well as their first order derivatives in the weak sense. The space $H^1(\mathbb{S}^{n-1})$ is implicitly endowed with the norm

$$\|h\|_{H^1} = \left(\|h\|_{L^2}^2 + \|\nabla h\|_{L^2}^2\right)^{1/2} = \left(\int_{\mathbb{S}^{n-1}} h^2 + \|\nabla h\|^2\right)^{1/2}$$

where the gradient ∇ is the one of the round sphere.

Let us recall basic facts about the Laplace–Beltrami operator Δ on $\mathbb{S}^{n-1}$. A reference for the results mentioned here is [12]. The first eigenvalues are denoted as follows:

$$0 = \lambda_0 < \lambda_1 = n-1 < \lambda_2 < \cdots .$$

The eigenspace associated to λ_0 is the space of constant functions. We will denote by $H^1(\mathbb{S}^{n-1})_0$ the subspace of $H^1(\mathbb{S}^{n-1})$ of functions L^2-orthogonal to the constant functions, i.e.

$$H^1(\mathbb{S}^{n-1})_0 = \{h \in H^1(\mathbb{S}^{n-1}) | \, (h,1)_{L^2} = 0\} ,$$

where $(\cdot,\cdot)_{L^2}$ is the L^2 scalar product on $\mathbb{S}^{n-1}$, and 1 is the constant function equal to 1 on $\mathbb{S}^{n-1}$. Obviously, this is also the space of functions in $H^1(\mathbb{S}^{n-1})$ which are H^1-orthogonal to the constant functions.

The eigenspace associated to λ_1 is the vector space spanned by the restrictions to the sphere of the coordinates functions of $\mathbb{R}^n$ (we identify the round sphere $\mathbb{S}^{n-1}$ with the sphere of unit vectors in the Euclidean space $\mathbb{R}^n$). We will denote by $H^1(\mathbb{S}^{n-1})_1$ the subspace of $H^1(\mathbb{S}^{n-1})$ of functions L^2-orthogonal to the eigenspace of λ_1, i.e.

$$\begin{aligned} H^1(\mathbb{S}^{n-1})_1 &= \{h \in H^1(\mathbb{S}^{n-1}) | \left(h, x^i\right)_{L^2} = 0 \ , i = 1,\ldots,n\} \\ &= \{h \in H^1(\mathbb{S}^{n-1}) | \int_{\mathbb{S}^{n-1}} h(x) x \,\mathrm{d}\mathbb{S}^{n-1}(x) = 0\} . \end{aligned}$$

Note that this is also the space of functions H^1-orthogonal to the vector space spanned by the restrictions to the sphere of the coordinates functions of $\mathbb{R}^n$. Indeed, for $i \in \{1,\ldots,n\}$ we have $\Delta x^i = \lambda_1 x^i$, hence $\int hx^i = 0$ implies $\int h\Delta x^i = 0$, and by Stoke's theorem $\int \langle \nabla h, \nabla x^i\rangle = 0$. Let us denote

$$H^1(\mathbb{S}^{n-1})_{01} = H^1(\mathbb{S}^{n-1})_0 \cap H^1(\mathbb{S}^{n-1})_1 .$$

By Rayleigh theorem, for $h \in H^1(\mathbb{S}^{n-1})_{01} \setminus \{0\}$ we have

$$\lambda_2 \leq \frac{\|\nabla h\|_{L^2}^2}{\|h\|_{L^2}^2} . \tag{2.1}$$

2.2 The form $\overline{V_2}^n$

We are interested in the following quadratic form on $H^1(\mathbb{S}^{n-1})$, which will be related to the intrinsic area V_2 in Section 4.2. For $h \in H^1(\mathbb{S}^{n-1})$,

$$\overline{V_2}^n(h) = c_n \left(\|h\|_{L^2}^2 - \lambda_1^{-1} \|\nabla h\|_{L^2}^2 \right) , \tag{2.2}$$

where c_n is a positive dimensional constant (recall that $\lambda_1 = n-1$). Its precise value will be relevant only in Section 2.4. This quadratic form comes from the following bilinear form $\overline{V_2}^n(\cdot,\cdot)$: for $h,k \in H^1(\mathbb{S}^{n-1})$,

$$\overline{V_2}^n(h,k) = c_n \left((h,k)_{L^2} - \lambda_1^{-1} (\nabla h, \nabla k)_{L^2} \right) .$$

To avoid confusions, let us emphasis that

$$\overline{V_2}^n(h,h) = \overline{V_2}^n(h) .$$

Fact 2.1. *The kernel of $\overline{V_2}^n(\cdot,\cdot)$ on $H^1(\mathbb{S}^{n-1})$ is the eigenspace of λ_1. In turn, $\overline{V_2}^n$ is non-degenerate on $H^1(\mathbb{S}^{n-1})_1$.*

Proof. Let $h \in H^1(\mathbb{S}^{n-1})$. The function h belongs to the kernel of $\overline{V_2}^n(\cdot,\cdot)$ if and only if for any $k \in H^1(\mathbb{S}^{n-1})$ we have

$$\int_{\mathbb{S}^{n-1}} hk = \lambda_1^{-1} \int_{\mathbb{S}^{n-1}} \langle \nabla h, \nabla k \rangle .$$

By density of smooth functions on $\mathbb{S}^{n-1}$ for the H^1-norm and by Green's formula, this is equivalent to the following property: for any smooth function k on $\mathbb{S}^{n-1}$ we have

$$\int_{\mathbb{S}^{n-1}} hk = \lambda_1^{-1} \int_{\mathbb{S}^{n-1}} h\Delta k ,$$

and this means $h = \lambda_1^{-1}\Delta h$ in the weak (hence smooth) sense. □

In the following, we will consider the restriction of $\overline{V_2}^n$ to $H^1(\mathbb{S}^{n-1})_1$, on which it is a non-degenerate bilinear form. Furthermore, we will see that the restriction of $\overline{V_2}^n$ to $H^1(\mathbb{S}^{n-1})_{01}$ is negative definite.

Lemma 2.2. *For $h \in H^1(\mathbb{S}^{n-1})_{01}$,*

$$c_n \left(\frac{\lambda_2 - \lambda_1}{\lambda_1} \right) \|h\|_{L^2}^2 \leq -\overline{V_2}^n(h) \tag{2.3}$$

and

$$c_n \left(\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} \right) \|h\|_{H^1}^2 \leq -\overline{V_2}^n(h) \leq c_n \frac{1}{\lambda_1} \|h\|_{H^1}^2 . \tag{2.4}$$

Proof. (2.3) is immediate from (2.1), and the right-hand side inequality in (2.4) follows from

$$-\overline{V_2}^n(h) \leq c_n \lambda_1^{-1} \|\nabla h\|_{L^2}^2 \leq c_n \lambda_1^{-1} \|h\|_{H^1}^2 .$$

The left-hand side inequality in (2.4) follows by adding the two following inequalities: as $\lambda_2 > \lambda_1 = n-1 \geq 1$, (2.3) gives

$$c_n \frac{1}{\lambda_2} \left(\frac{\lambda_2 - \lambda_1}{\lambda_1} \right) \|h\|_{L^2}^2 \leq -\overline{V_2}^n(h) ,$$

and on the other hand, using again (2.1), the equality (2.2) gives

$$c_n \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) \|\nabla h\|_{L^2}^2 \leq -\overline{V_2}^n(h) .$$

□

Proposition 2.3. $(H^1(\mathbb{S}^{n-1})_{01}, -\overline{V_2}^n(\cdot,\cdot))$ *is a separable Hilbert space.*

Proof. By (2.3) or (2.4), $-\overline{V_2}^n$ is a scalar product on $H^1(\mathbb{S}^{n-1})_{01}$. As $H^1(\mathbb{S}^{n-1})_{01}$ is orthogonal to a vector subspace, it is a closed subspace, hence complete and separable for the H^1 norm. The result follows from (2.4).

□

Let L be the line of constant functions in $H^1(\mathbb{S}^{n-1})_1$. For $h \in H^1(\mathbb{S}^{n-1})_1$, let us introduce the number

$$\mathbf{mean}(h) = \frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} h \ .$$

Note that we have $\mathbf{mean}(h) = c_n^{-1}\,\mathrm{vol}(\mathbb{S}^{n-1})^{-1}\overline{V_2}^n(h,1)$. Abusing notation, we will denote by **mean** the function from $H^1(\mathbb{S}^{n-1})_1$ to L sending h to the function on the sphere having constant value $\mathbf{mean}(h)$. Obviously $H^1(\mathbb{S}^{n-1})_{01}$ is the set of elements h of $H^1(\mathbb{S}^{n-1})_1$ such that $\mathbf{mean}(h) = 0$, and from the definitions it is immediate that $H^1(\mathbb{S}^{n-1})_{01}$ and L are $\overline{V_2}^n$-orthogonal.

For future reference, note that for any $h \in H^1(\mathbb{S}^{n-1})_1$ we have $h - \mathbf{mean}(h) \in H^1(\mathbb{S}^{n-1})_{01}$, hence for any constant function c,

$$\overline{V_2}^n(h - \mathbf{mean}(h), c) = 0 \ .$$

For any $h, k \in H^1(\mathbb{S}^{n-1})_1$ that gives, writing $h = (h - \mathbf{mean}(h)) + \mathbf{mean}(h)$ and $k = (k - \mathbf{mean}(k)) + \mathbf{mean}(k)$,

$$\overline{V_2}^n(h,k) = \overline{V_2}^n(h - \mathbf{mean}(h), k - \mathbf{mean}(k)) + \mathbf{mean}(h)\mathbf{mean}(k)\overline{V_2}^n(1) \ . \tag{2.5}$$

2.3 Inequalities and signature

Let us introduce the following cone (see Figure 2)

$$\mathcal{C}_n = \{h \in H^1(\mathbb{S}^{n-1})_1 | \overline{V_2}^n(h) > 0, \mathbf{mean}(h) > 0\} \ , \tag{2.6}$$

and let

$$\overline{\mathcal{C}}_n = \{h \in H^1(\mathbb{S}^{n-1})_1 | \overline{V_2}^n(h) \geq 0, \mathbf{mean}(h) > 0\} \ .$$

Fact 2.4. *For any $h, k \in \mathcal{C}_n$ we have*

$$\overline{V_2}^n(h,k) > 0 \tag{2.7}$$

and for any $h, k \in \overline{\mathcal{C}}_n$ we have

$$\overline{V_2}^n(h,k) \geq 0 \ . \tag{2.8}$$

Proof. Notice that (2.8) is a consequence of (2.7): indeed, for every $h, k \in \overline{\mathcal{C}}_n$ and $\epsilon > 0$ we have $h + \epsilon, k + \epsilon \in \mathcal{C}_n$: we have

$$\overline{V_2}^n(h+\epsilon) = \overline{V_2}^n(h) + 2\epsilon\overline{V_2}^n(h,1) + \epsilon^2\overline{V_2}^n(1) > 0$$

(remember that $\overline{V_2}^n(h,1) = c_n\,\mathrm{vol}(\mathbb{S}^{n-1})\mathbf{mean}(h) > 0$), and the same holds for $\overline{V_2}^n(k+\epsilon)$. So if (2.7) is true we have

$$\overline{V_2}^n(h+\epsilon, k+\epsilon) = \overline{V_2}^n(h,k) + \epsilon\overline{V_2}^n(h,1) + \epsilon\overline{V_2}^n(1,k) + \epsilon^2\overline{V_2}^n(1) > 0 \ ,$$

and when ϵ goes to zero this gives (2.8).

Now, let us prove (2.7). If $h \in \mathcal{C}_n$, then $\overline{V_2}^n(h) > 0$, hence by (2.5),

$$0 < \overline{V_2}^n(h) = \overline{V_2}^n(h - \mathbf{mean}(h)) + \mathbf{mean}(h)^2\overline{V_2}^n(1) \ .$$

As $-\overline{V_2}^n$ is non-negative on $H^1(\mathbb{S}^{n-1})_{01}$ (Lemma 2.2), this gives

$$0 \leq -\overline{V_2}^n(h - \mathbf{mean}(h)) < \mathbf{mean}(h)^2\overline{V_2}^n(1) \ .$$

Using the classic Cauchy–Schwarz inequality in $H^1(\mathbb{S}^{n-1})_{01}$ (Proposition 2.3) and the above equation we obtain

$$\begin{aligned} -\overline{V_2}^n(h - \mathbf{mean}(h), k - \mathbf{mean}(k)) &\leq \sqrt{-\overline{V_2}^n(h - \mathbf{mean}(h))}\sqrt{-\overline{V_2}^n(k - \mathbf{mean}(k))} \\ &< \mathbf{mean}(h)\mathbf{mean}(k)\overline{V_2}^n(1) \end{aligned}$$

(recall that $\mathbf{mean}(h)$ and $\mathbf{mean}(k)$ are both positive). The result follows from (2.5). □

The following is an immediate consequence of (2.7).

Fact 2.5. *The cone $\mathcal{C}_n$ is convex.*

The following fact explains why $(H^1(\mathbb{S}^{n-1})_1, \overline{V_2}^n)$ may be considered as an infinite dimensional Minkowski space (in the Lorentzian sense).

Fact 2.6. *The restriction of $\overline{V_2}^n(\cdot,\cdot)$ to any vector subspace of finite dimension p of $H^1(\mathbb{S}^{n-1})_1$, containing an element of $\mathcal{C}_n$, has Lorentzian signature $(+,-,\dots,-)$, with 1 positive direction and $p-1$ negative directions.*

Proof. By definition, $\overline{V_2}^n$ is positive when evaluated on any vector of $\mathcal{C}_n$, and the intersection of the vector space with $H^1(\mathbb{S}^{n-1})_{01}$ has dimension $(p-1)$, on which $\overline{V_2}^n$ is negative definite from Proposition 2.3. □

Fact 2.7 (Reversed Cauchy–Schwarz inequality)**.** *For any $h, k \in \overline{\mathcal{C}}_n$, we have*

$$\overline{V_2}^n(h,k)^2 \geq \overline{V_2}^n(h)\overline{V_2}^n(k) \ , \tag{2.9}$$

or if one prefers, by Fact 2.4,

$$\overline{V_2}^n(h,k) \geq \overline{V_2}^n(h)^{1/2}\overline{V_2}^n(k)^{1/2} \ . \tag{2.10}$$

Equality holds if and only if $h = \lambda k$, $\lambda > 0$.

Proof. The equality is obvious if $h, k \in \overline{\mathcal{C}}_n$ are colinear.

Let $h, k \in \mathcal{C}_n$ be non-colinear. Then they span a plane, which, by Fact 2.6, contains a positive and a negative vector for $\overline{V_2}^n$. So there exists a t such that $\overline{V_2}^n(h+tk)$ is positive, and a t such that $\overline{V_2}^n(h+tk)$ is negative, hence the second order polynomial $t \mapsto \overline{V_2}^n(h+tk)$ has a positive discriminant. This gives (2.9) with a strict inequality.

Now let $h, k \in \overline{\mathcal{C}}_n$ be non-colinear. If $h, k \in \mathcal{C}_n$ then we know that (2.9) holds with a strict inequality, so for example assume that $\overline{V_2}^n(k) = 0$. Then $\overline{V_2}^n(h+tk) = \overline{V_2}^n(h) + 2t\overline{V_2}^n(h,k)$, and h and k span a plane, hence there exists some t such that $\overline{V_2}^n(h+tk) < 0$. This shows that $\overline{V_2}^n(h,k) \neq 0$, hence $\overline{V_2}^n(h,k)^2 > 0 = \overline{V_2}^n(h)\overline{V_2}^n(k)$. □

2.4 Intrinsic nature of $\overline{V_1}^n$ and $\overline{V_2}^n$

We settle that, in the definition of $\overline{V_2}^n$,

$$c_n = \frac{(n-1)W_{n-1}}{\kappa_{n-1}} \overset{(1.3)}{=} \frac{n-1}{2\kappa_{n-2}} \tag{2.11}$$

(recall that W_n is the Wallis' integral). For future reference, let us note that in particular we have $c_2 = 1/2$, and that the dimensional constant $r_2(n)$ introduced in (1.4) satisfies

$$r_2(n) = \overline{V_2}^n(1)^{-1/2} \ .$$

This definition of c_n allows to prove that, for natural embeddings of $H^1(\mathbb{S}^{n-1})$ into $H^1(\mathbb{S}^n)$ (defined below), the corresponding value of $\overline{V_2}^n$ will not change.

In $\mathbb{R}^{n+1}$, choose a point $N \in \mathbb{S}^n$, and choose a direct orthonormal base $\mathcal{B}$ of the hyperplane $N^\perp$, such that $(\mathcal{B}, N)$ is a direct orthonormal basis of $\mathbb{R}^{n+1}$. Let $R_{\mathcal{B},N} \in SO(n+1)$ be the unique rotation sending the canonical basis of $\mathbb{R}^{n+1}$ on $(\mathcal{B}, N)$. It is clear that $R_{\mathcal{B},N}$, restricted to $\mathbb{S}^{n-1} \equiv \mathbb{S}^{n-1} \times \{0\} \subset \mathbb{R}^{n+1}$ is an isometric embedding of $\mathbb{S}^{n-1}$ into $\mathbb{S}^n$ (given by the intersection $N^\perp \cap \mathbb{S}^n$). Moreover, every such embedding is obtained in this way.

This gives the following coordinates on $\mathbb{S}^n \setminus \{N, -N\}$:

$$\Phi_{\mathcal{B},N} : \mathbb{S}^{n-1} \times]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{S}^n,\ (x,t) \mapsto R_{\mathcal{B},N}(\cos(t)x, \sin(t)) \ .$$

Remark 2.8. Assume that $R_{\mathcal{B},N}$ is the identity, and let $\Phi = \Phi_{\mathcal{B},N}$. Then an orthonormal basis of the tangent space of $\mathbb{S}^n$ at the point $\Phi(x,t)$ is

$$\left(\frac{1}{\cos(t)} \nabla_{\partial\theta_1}\Phi(x,t), \ldots, \frac{1}{\cos(t)} \nabla_{\partial\theta_{n-1}}\Phi(x,t), \nabla_{\partial t}\Phi(x,t) \right) \ ,$$

where $\partial\theta_1, \ldots, \partial\theta_{n-1}$ is an orthonormal basis of the tangent space of $\mathbb{S}^{n-1}$ at the point x. In particular, the Jacobian of Φ at the point (x,t) is $\cos^{n-1}(t)$.

For $h \in H^1(\mathbb{S}^{n-1})$ let us define $E_{\mathcal{B},N}(h) : \mathbb{S}^n \to \mathbb{R}$ by 0 on $\{N, -N\}$, and otherwise

$$E_{\mathcal{B},N}(h) \circ \Phi_{\mathcal{B},N}(x,t) = \cos(t) h(x) \ .$$

The geometric meaning of the functions $E_{\mathcal{B},N}(h)$ will be clarified in Section 4.1 and Fact 4.10.

Proposition 2.9. *We have defined an injective linear map $E_{\mathcal{B},N}$ from $H^1(\mathbb{S}^{n-1})$ to $H^1(\mathbb{S}^n)$. Moreover, $E_{\mathcal{B},N}(H^1(\mathbb{S}^{n-1})_1) \subset H^1(\mathbb{S}^n)_1$ and $E_{\mathcal{B},N}(H^1(\mathbb{S}^{n-1})_{01}) \subset H^1(\mathbb{S}^n)_{01}$.*

The first part of the proposition is obvious, and the two inclusions are direct consequences of the following lemma.

Lemma 2.10. *For $h \in H^1(\mathbb{S}^{n-1})$ we have*

$$\int_{\mathbb{S}^n} E_{\mathcal{B},N}(h) = 2W_n \int_{\mathbb{S}^{n-1}} h \tag{2.12}$$

and

$$\int_{\mathbb{S}^n} E_{\mathcal{B},N}(h)(v)v = R_{\mathcal{B},N}\left(2W_{n+1} \int_{\mathbb{S}^{n-1}} h(u)u, 0\right) \in \mathbb{R}^{n+1} \ .$$

Proof. The Jacobian of Φ_N at the point (x,t) is $\cos^{n-1}(t)$, hence a change of variable gives the first identity.

For the second one, with the same change of variable we obtain

$$\int_{\mathbb{S}^n} E_{\mathcal{B},N}(h)(v)v = \int_{\mathbb{S}^{n-1}\times]-\frac{\pi}{2},\frac{\pi}{2}[} \cos^n(t) h(x) R_{\mathcal{B},N}(\cos(t)x, \sin(t)) \ ,$$

and this gives the result since $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(t)\sin(t)\mathrm{d}t = 0$. □

Let us define

$$\overline{V_1}^n(h) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} h \ .$$

Note that $\overline{V_1}^n(h) = \frac{n\kappa_n}{\kappa_{n-1}}\textbf{mean}(h)$. As we will show later, this quantity is related to $V_1(K)$ for a convex body K. As $2W_n = \frac{\kappa_n}{\kappa_{n-1}}$, the first formula in Lemma 2.10 says that

$$\overline{V_1}^{n+1}(E_{\mathcal{B},N}(h)) = \overline{V_1}^n(h) \ . \tag{2.13}$$

Proposition 2.11. *Let $h \in H^1(\mathbb{S}^{n-1})$. The quantity $\overline{V_2}^n(h)$ is intrinsic in the following sense:*

$$\overline{V_2}^{n+1}(E_{\mathcal{B},N}(h)) = \overline{V_2}^n(h)\ .$$

Proof. Since the integral of h^2 and $\|\nabla h\|^2$ are invariant by rotation, we may assume that $R_{\mathcal{B},N}$ is the identity (that is, the embedding of $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$ is given by $\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \{0\}$). For simplicity, let $E = E_{\mathcal{B},N}$ and $\Phi = \Phi_{\mathcal{B},N}$.

We have the relation

$$\|\nabla E(h)(\Phi(x,t))\|^2 = \|\nabla h(x)\|^2 + \sin^2(t)h(x)^2\ .$$

Indeed, using the notations of Remark 2.8, the relation $E(h) \circ \Phi(x,t) = \cos(t)h(x)$ and the chain rule formula we get

$$\nabla E(h)(\Phi(x,t)) \cdot \left(\frac{1}{\cos(t)}\nabla_{\partial\theta_i}\Phi(x,t)\right) = \frac{1}{\cos(t)}\nabla_{\partial\theta_i}(E(h)\circ\Phi)(x,t) = \nabla_{\partial\theta_i}h(x)$$

and

$$\nabla E(h)(\Phi(x,t)) \cdot \nabla_{\partial t}\Phi(x,t) = \nabla_{\partial t}(E(h)\circ\Phi)(x,t) = -\sin(t)h(x)\ .$$

Hence by the change of variables given by Φ we obtain

$$\begin{aligned}\int_{\mathbb{S}^n}\|\nabla E(h)\|^2 &= \int_{\mathbb{S}^{n-1}\times]-\frac{\pi}{2},\frac{\pi}{2}[}\cos^{n-1}(t)\left(\|\nabla h\|^2 + \sin^2(t)h^2\right)\\ &= 2W_{n-1}\int_{\mathbb{S}^{n-1}}\|\nabla h\|^2 + 2(W_{n-1}-W_{n+1})\int_{\mathbb{S}^{n-1}}h^2\ ,\end{aligned}$$

and by the same change of variables we get

$$\int_{\mathbb{S}^n}E(h)^2 = \int_{\mathbb{S}^{n-1}\times]-\frac{\pi}{2},\frac{\pi}{2}[}\cos^{n-1}(t)\cos^2(t)h^2 = 2W_{n+1}\int_{\mathbb{S}^{n-1}}h^2\ .$$

This gives

$$\begin{aligned}\overline{V_2}^{n+1}(E(h)) &= c_{n+1}\left(\int_{\mathbb{S}^n}E(h)^2 - \frac{1}{n}\int_{\mathbb{S}^n}\|\nabla E(h)\|^2\right)\\ &= 2c_{n+1}\left(\left(W_{n+1} - \frac{W_{n-1}-W_{n+1}}{n}\right)\int_{\mathbb{S}^{n-1}}h^2 - \frac{W_{n-1}}{n}\int_{\mathbb{S}^{n-1}}\|\nabla h\|^2\right)\ .\end{aligned}$$

Using the relation $W_{n+1} = \frac{n}{n+1}W_{n-1}$ we get

$$W_{n+1} - \frac{W_{n-1}-W_{n+1}}{n} = \frac{n+1}{n}W_{n+1} - \frac{1}{n}W_{n-1} = (n-1)\frac{W_{n-1}}{n}\ ,$$

and noting that $\frac{2c_{n+1}W_{n-1}(n-1)}{n} = c_n$ for any $n > 1$ (see equation (1.3)), we finally obtain

$$\overline{V_2}^{n+1}(E(h)) = \frac{2c_{n+1}W_{n-1}(n-1)}{n}\left(\int_{\mathbb{S}^{n-1}}h^2 - \frac{1}{n-1}\int_{\mathbb{S}^{n-1}}\|\nabla h\|^2\right) = \overline{V_2}^n(h)\ .$$

□

3 Infinite dimensional hyperbolic space

In this section, we use the bilinear form $\overline{V_2}^n$ defined in the preceding section, in order to define an infinite dimensional hyperbolic space. The content of this section is almost elementary. Actually, apart from some convergence properties, the constructions are formally the same as in the finite dimensional case. However we are provinding details for the sake of completeness.

We introduce three models of the infinite dimensional hyperbolic space:

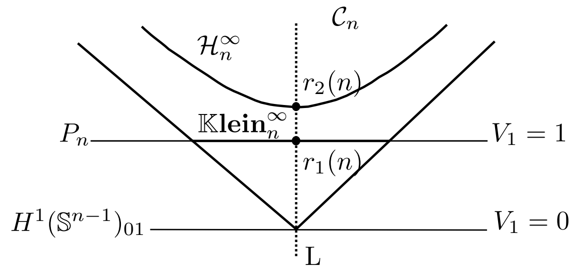

Figure 2: Notations for subspaces of $H^1(\mathbb{S}^{n-1})_1$.

- a hyperboloid model $\mathcal{H}_n^\infty$ in $(H^1(\mathbb{S}^{n-1})_1, \overline{V_2}^n)$, which is in natural bijection with $H^1(\mathbb{S}^{n-1})_{01}$, that will allow to check some completeness properties;
- a projective model $\mathbb{H}_n^\infty$, which allows to get rid of normalizations, and on which we can define a Hilbert distance;
- a Klein ball model $\mathbb{K}\mathbf{lein}_n^\infty$, an affine chart of the projective model, in which the geodesics are very easy to describe (these are convex combinations).

3.1 Hyperboloid model

Let us introduce

$$\mathcal{H}_n^\infty = \{h \in \mathcal{C}_n | \overline{V_2}^n(h) = 1\} \ ,$$

where $\mathcal{C}_n$ is the cone defined in (2.6).

Let $\mathbf{p}_\mathcal{H} : \mathcal{H}_n^\infty \to H^1(\mathbb{S}^{n-1})_{01}$ be the map $\mathbf{p}_\mathcal{H}(h) = h - \mathbf{mean}(h)$. It is a bijection between $\mathcal{H}_n^\infty$ and $H^1(\mathbb{S}^{n-1})_{01}$, and its inverse is the map $\mathbf{p}_\mathcal{H}^{-1} : H^1(\mathbb{S}^{n-1})_{01} \to \mathcal{H}_n^\infty$ defined by $\mathbf{p}_\mathcal{H}^{-1}(h) = h + \mathbf{c}(h)$, where $\mathbf{c}(h)$ is the constant function on $\mathbb{S}^{n-1}$ defined as

$$\mathbf{c}(h) = r_2(n)\sqrt{1 - \overline{V_2}^n(h)} \ :$$

indeed, since $h \in H^1(\mathbb{S}^{n-1})_{01}$, we have $\overline{V_2}^n(h + \mathbf{c}(h)) = \overline{V_2}^n(h) + \mathbf{c}(h)^2\overline{V_2}^n(1)$, hence the equation $\overline{V_2}^n(h + \mathbf{c}(h)) = 1$ gives the formula above (recall that $r_2(n) = \overline{V_2}^n(1)^{-1/2}$). Note that $\mathbf{c}(h)$ is well defined on $H^1(\mathbb{S}^{n-1})_{01}$, on which $\overline{V_2}^n \leq 0$.

As the Hilbert structure on $H^1(\mathbb{S}^{n-1})_{01}$ is given by $\overline{V_2}^n$, the map $\overline{V_2}^n$ is smooth, so $\mathbf{c} : H^1(\mathbb{S}^{n-1})_{01} \to \mathbb{R}$ is also smooth. It follows that as a graph, $\mathcal{H}_n^\infty$ is an infinite dimensional smooth manifold. We implicitly endow it with the restriction of $-\overline{V_2}^n(\cdot, \cdot)$ on its tangent spaces. The intersection of $\mathcal{H}_n^\infty$ with any vector p-plane of $H^1(\mathbb{S}^{n-1})_1$ containing a vector of $\mathcal{C}_n$ is a hyperboloid model of the hyperbolic space of dimension $(p-1)$ by Fact 2.6. In turn, $\mathcal{H}_n^\infty$ is a Riemannian manifold of constant sectional curvature -1. However, it will deserve the name *infinite dimensional hyperbolic space* once we will have checked its completeness, that is the content of the remainder of this section, see Corollary 3.6.

At a metric level, Fact 2.7 implies that for any $h, k \in \mathcal{H}_n^\infty$ we have $\overline{V_2}^n(h, k) \geq 1$, so we can define the following: for $h, k \in \mathcal{H}_n^\infty$ we set

$$d_\mathcal{H}(h, k) = \operatorname{argch} \overline{V_2}^n(h, k) \ . \tag{3.1}$$

Using again an intersection with a finite dimensional vector space, it is well known that $d_\mathcal{H}$ is indeed a distance, actually the one given by the Riemannian structure. In the same way we obtain the following.

Fact 3.1. *$(\mathcal{H}_n^\infty, d_{\mathcal{H}})$ is a uniquely geodesic metric space, and the shortest path between h and k is the intersection of $\mathcal{H}_n^\infty$ with the vector plane spanned by h and k.*

We will denote by d_{01} the distance induced by $\sqrt{-\overline{V_2}^n}$ on $H^1(\mathbb{S}^{n-1})_{01}$: for $h,k \in H^1(\mathbb{S}^{n-1})_{01}$ we set

$$d_{01}(h,k) = \sqrt{-\overline{V_2}^n(h-k)} \ .$$

Lemma 3.2. *The map $\mathbf{p}_{\mathcal{H}} : \mathcal{H}_n^\infty \to H^1(\mathbb{S}^{n-1})_{01}$ is expanding: for every $h,k \in \mathcal{H}_n^\infty$ we have*

$$d_{\mathcal{H}}(h,k) \leq d_{01}(\mathbf{p}_{\mathcal{H}}(h), \mathbf{p}_{\mathcal{H}}(k)) \ . \tag{3.2}$$

Proof. For $h,k \in \mathcal{H}_n^\infty$, the inequality $\overline{V_2}^n(h,k) \geq 1 + d_{\mathcal{H}}(h,k)^2/2$ follows directly from $\cosh(u) \geq 1 + u^2/2$ and (3.1), and with $\overline{V_2}^n(h-k) = 2 - 2\overline{V_2}^n(h,k)$ we obtain

$$d_{\mathcal{H}}(h,k) \leq \sqrt{-\overline{V_2}^n(h-k)} \ .$$

Moreover by (2.5), for every $f \in H^1(\mathbb{S}^{n-1})_1$,

$$-\overline{V_2}^n(f - \mathbf{mean}(f)) = -\overline{V_2}^n(f) + \mathbf{mean}(f)^2 \overline{V_2}^n(1) \geq -\overline{V_2}^n(f) \ ,$$

hence

$$\sqrt{-\overline{V_2}^n(\mathbf{p}_{\mathcal{H}}(h) - \mathbf{p}_{\mathcal{H}}(k))} = \sqrt{-\overline{V_2}^n(h - k - \mathbf{mean}(h-k)))} \geq \sqrt{-\overline{V_2}^n(h-k)}$$

and this gives the result. □

Lemma 3.3. *The map $\mathbf{c}$ is $(r_2(n)\tanh(t))$-Lipschitz on the ball of radius $\sinh(t)$ centered at 0 in $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$.*

Proof. For $u,v \geq 0$ we have the formula $\sqrt{1+u} - \sqrt{1+v} = \frac{\sqrt{u}+\sqrt{v}}{\sqrt{1+u}+\sqrt{1+v}}(\sqrt{u} - \sqrt{v})$, and this gives

$$\begin{aligned} |\mathbf{c}(h) - \mathbf{c}(k)| &= r_2(n) \left| \sqrt{1 - \overline{V_2}^n(h)} - \sqrt{1 - \overline{V_2}^n(k)} \right| \\ &= r_2(n) \frac{\sqrt{-\overline{V_2}^n(h)} + \sqrt{-\overline{V_2}^n(k)}}{\sqrt{1 - \overline{V_2}^n(h)} + \sqrt{1 - \overline{V_2}^n(k)}} \left| \sqrt{-\overline{V_2}^n(h)} - \sqrt{-\overline{V_2}^n(k)} \right| \ . \end{aligned}$$

Moreover, $\left| \sqrt{-\overline{V_2}^n(h)} - \sqrt{-\overline{V_2}^n(k)} \right| = |d_{01}(0,h) - d_{01}(0,k)| \leq d_{01}(h,k)$, and there exists $s, s' < t$ such that $\sqrt{-\overline{V_2}^n(h)} = \sinh(s)$ and $\sqrt{-\overline{V_2}^n(k)} = \sinh(s')$, hence $\frac{\sqrt{-\overline{V_2}^n(h)}+\sqrt{-\overline{V_2}^n(k)}}{\sqrt{1-\overline{V_2}^n(h)}+\sqrt{1-\overline{V_2}^n(k)}} = \tanh(\frac{s+s'}{2}) \leq \tanh(t)$.

□

Lemma 3.4. *Let O be the ball centered at 0 and of radius $\sinh(t)$ in $H^1(\mathbb{S}^{n-1})_{01}$. For every $h,k \in \mathbf{p}_{\mathcal{H}}^{-1}(O) \subset \mathcal{H}_n^\infty$, we have*

$$(1 - \tanh(t)) d_{01}(\mathbf{p}_{\mathcal{H}}(h), \mathbf{p}_{\mathcal{H}}(k)) \leq d_{\mathcal{H}}(h,k) \ . \tag{3.3}$$

Proof. Let $\gamma : [0, d_{\mathcal{H}}(h,k)] \to \mathcal{H}_n^\infty$ be the arc-length parametrized geodesic between h and k in $\mathcal{H}_n^\infty$. We have $d_{\mathcal{H}}(h,k) = \int \sqrt{-\overline{V_2}^n((\mathbf{p}_{\mathcal{H}}^{-1} \circ \bar{\gamma})')}$, with $\bar{\gamma} = \mathbf{p}_{\mathcal{H}}(\gamma)$. Note that $(\mathbf{p}_{\mathcal{H}}^{-1} \circ \bar{\gamma})' =$

$(\bar\gamma + c\circ\bar\gamma)' = \bar\gamma' + (c\circ\bar\gamma)'$. This gives, using the inequality $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ valid for $a \geq b \geq 0$,

$$\sqrt{-\overline{V_2}^n((\mathbf{p}_{\mathcal{H}}^{-1}\circ\bar\gamma)')} = \sqrt{-\overline{V_2}^n(\bar\gamma') - (c\circ\bar\gamma)'^2 r_2(n)^{-2}} \geq \sqrt{-\overline{V_2}^n(\bar\gamma')} - |(c\circ\bar\gamma)'| r_2(n)^{-1}$$

(recall that $\overline{V_2}^n(1) = r_2(n)^{-2}$). By Lemma 3.3, $|(c\circ\bar\gamma)'| \leq r_2(n)\tanh(t)\sqrt{-\overline{V_2}^n(\bar\gamma')}$. Finally we obtain

$$d_{\mathcal{H}}(h,k) \geq (1-\tanh(t))\int \sqrt{-\overline{V_2}^n(\bar\gamma')}\ ,$$

and since $\int \sqrt{-\overline{V_2}^n(\bar\gamma')}$ is the length of a curve in $H^1(\mathbb{S}^{n-1})_{01}$ between $\mathbf{p}_{\mathcal{H}}(h)$ and $\mathbf{p}_{\mathcal{H}}(k)$, it is $\geq d_{01}(\mathbf{p}_{\mathcal{H}}(h), \mathbf{p}_{\mathcal{H}}(k))$ and this ends the proof. □

Lemma 3.2 and 3.4 give the following.

Corollary 3.5. *The map* $\mathbf{p}_{\mathcal{H}} : (\mathcal{H}_n^\infty, d_{\mathcal{H}}) \to (H^1(\mathbb{S}^{n-1})_{01}, d_{01})$ *is locally bi-Lipschitz.*

Hence from Proposition 2.3 one obtains the following.

Corollary 3.6. $(\mathcal{H}_n^\infty, d_{\mathcal{H}})$ *is complete.*

We also obtain the following.

Corollary 3.7. *On* $\mathcal{H}_n^\infty$*,* $d_{\mathcal{H}}$ *and* $\|\cdot\|_{H^1}$ *induce the same topology.*

Proof. Let (h_i) be a sequence in $\mathcal{H}_n^\infty$ converging to $h \in \mathcal{H}_n^\infty$ for $\|\cdot\|_{H^1}$. Then (h_i) (resp. (∇h_i)) converges to h (resp. ∇h) in L^2, hence $\overline{V_2}^n(h_i - h) \to 0$. But $\overline{V_2}^n(h_i - h) = 2 - 2\overline{V_2}^n(h_i, h)$, hence $\overline{V_2}^n(h_i, h) \to 1$ and $d_{\mathcal{H}}(h_i, h) = \mathrm{argch}(\overline{V_2}^n(h_i, h)) \to 0$.

On the other hand, let (h_i) be a sequence in $\mathcal{H}_n^\infty$ converging to $h \in \mathcal{H}_n^\infty$ for $d_{\mathcal{H}}$. Then by Corollary 3.5 we have $\mathbf{p}_{\mathcal{H}}(h_i) = h_i - \mathbf{mean}(h_i) \to \mathbf{p}_{\mathcal{H}}(h) = h - \mathbf{mean}(h)$ in $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$, hence $\nabla\mathbf{p}_{\mathcal{H}}(h_i) \to \nabla\mathbf{p}_{\mathcal{H}}(h)$ in L^2 (see Lemma 2.2), so $\nabla h_i \to \nabla h$ in L^2. Moreover, $\overline{V_2}^n(h - h_i) = c_n(\|h - h_i\|_{L^2}^2 - \lambda_1^{-1}\|\nabla h - \nabla h_i\|_{L^2}^2) \to 0$, hence $h_i \to h$ in L^2, and this gives $h_i \to h$ for $\|\cdot\|_{H^1}$. □

Remark 3.8. From Proposition 2.11, the maps $E_{\mathcal{B},N}$ induce totally geodesic isometric immersions of $\mathcal{H}_n^\infty$ into $\mathcal{H}_{n+1}^\infty$.

Remark 3.9. As in the finite dimensional case, the isometry group of $(\mathcal{H}_n^\infty, d_{\mathcal{H}})$ is given by the linear maps preserving $\overline{V_2}^n$, up to a quotient by $\pm\mathrm{Id}$. We refer to [11] for this fact and others about isometries of the infinite dimensional hyperbolic space.

3.2 Projective model

Let $h, k \in \mathcal{C}_n$, with $h \neq k$, be such that $\overline{V_2}^n(h) = \overline{V_2}^n(k) = a > 0$. Consider the equation

$$\overline{V_2}^n((1-t)h + tk) = 0$$

(see also equation (1.8) in the introduction). The left-hand side gives the polynomial

$$t^2\overline{V_2}^n(h-k) + 2t\overline{V_2}^n(h, k-h) + \overline{V_2}^n(h)\ .$$

Since $\overline{V_2}^n(h) = \overline{V_2}^n(k) = a$, we have

$$\overline{V_2}^n(h-k) = 2\left(a - \overline{V_2}^n(h,k)\right)\ ,$$

and by Fact 2.7 this is negative (remember that $h \neq k$): hence this is a second order polynomial. A direct computation shows that the discriminant is $4\delta(h,k)$, where we set

$$\delta(h,k) = \overline{V_2}^n(h,k)^2 - a^2 > 0 \ .$$

The two real solutions are $t_1 < 0 < 1 < t_2$ with

$$t_1 = \frac{\overline{V_2}^n(h,h-k) - \sqrt{\delta(h,k)}}{\overline{V_2}^n(h-k)} \text{ and } t_2 = \frac{\overline{V_2}^n(h,h-k) + \sqrt{\delta(h,k)}}{\overline{V_2}^n(h-k)} \ .$$

Let us denote by $\mathbb{H}_n^\infty$ the projective quotient of $\mathcal{C}_n$, and, for $[h],[k] \in \mathbb{H}_n^\infty$, let us define

$$d_{\mathbb{H}}([h],[k]) := \frac{1}{2}\ln[0,1,t_1,t_2] \ ,$$

where h,k are representative of $[h],[k]$ respectively, such that $\overline{V_2}^n(h) = \overline{V_2}^n(k)$, and $[0,1,t_1,t_2] = \frac{t_1}{t_2}\frac{1-t_2}{1-t_1}$ is the cross-ratio.

Fact 3.10. *For $h,k \in \mathcal{C}_n$ we have*

$$d_{\mathbb{H}}([h],[k]) = d_{\mathcal{H}}(\overline{V_2}^n(h)^{-1/2}h, \overline{V_2}^n(k)^{-1/2}k) = \operatorname{argch}\left(\frac{\overline{V_2}^n(h,k)}{\sqrt{\overline{V_2}^n(h)\overline{V_2}^n(k)}}\right) \ . \qquad (3.4)$$

Proof. Let $a = \overline{V_2}^n(h) = \overline{V_2}^n(k)$. A direct computation yields to $t_1 t_2 = \frac{\overline{V_2}^n(h)}{\overline{V_2}^n(h-k)}$, and this gives

$$[0,1,t_1,t_2] = \frac{t_1 - t_1t_2}{t_2 - t_1t_2} = \frac{\overline{V_2}^n(h,h-k) - \sqrt{\delta(h,k)} - \overline{V_2}^n(h)}{\overline{V_2}^n(h,h-k) + \sqrt{\delta(h,k)} - \overline{V_2}^n(h)} = \frac{\overline{V_2}^n(h,k) + \sqrt{\delta(h,k)}}{\overline{V_2}^n(h,k) - \sqrt{\delta(h,k)}} \ ,$$

hence

$$\begin{aligned} d_{\mathbb{H}}([h],[k]) &= \frac{1}{2}\ln\left(\frac{\overline{V_2}^n(h,k) + \sqrt{\delta(h,k)}}{\overline{V_2}^n(h,k) - \sqrt{\delta(h,k)}}\right) = \frac{1}{2}\ln\left(\frac{\left(\overline{V_2}^n(h,k) + \sqrt{\delta(h,k)}\right)^2}{\overline{V_2}^n(h,k)^2 - \delta(h,k)}\right) \\ &= \frac{1}{2}\ln\left(\frac{\left(\overline{V_2}^n(h,k) + \sqrt{\delta(h,k)}\right)^2}{a^2}\right) = \ln\left(\frac{\overline{V_2}^n(h,k)}{a} + \frac{\sqrt{\delta(h,k)}}{a}\right) \ . \end{aligned}$$

And the formula $\operatorname{argch}(u) = \ln(u + \sqrt{u^2-1})$ valid for $u \geq 1$ gives

$$\operatorname{argch}\left(\frac{\overline{V_2}^n(h,k)}{a}\right) = \ln\left(\frac{\overline{V_2}^n(h,k)}{a} + \sqrt{\frac{\overline{V_2}^n(h,k)^2}{a^2} - 1}\right) = \ln\left(\frac{\overline{V_2}^n(h,k)}{a} + \frac{\sqrt{\delta(h,k)}}{a}\right)$$

and this ends the proof. □

Hence the map

$$\mathrm{Nor}_{\overline{V_2}^n} : \mathbb{H}_n^\infty \to \mathcal{H}_n^\infty \ ,$$

defined by choosing the unique representative of an equivalence class in $\mathbb{H}_n^\infty$ which is in $\mathcal{H}_n^\infty$ (that is, which satisfies $\overline{V_2}^n = 1$), is an isometry between $(\mathbb{H}_n^\infty, d_{\mathbb{H}})$ and $(\mathcal{H}_n^\infty, d_{\mathcal{H}})$.

3.3 Klein ball model

It is useful to consider an affine model of $(\mathbb{H}_n^\infty, d_{\mathbb{H}})$. Let us consider the intersection $\mathbb{K}\mathbf{lein}_n^\infty$ of $\mathcal{C}_n$ with the affine hyperplane

$$P_n = \{h \in H^1(\mathbb{S}^{n-1})_1 \mid \overline{V_1}^n(h) = 1\}$$

(see Figure 2), that is

$$\mathbb{K}\mathbf{lein}_n^\infty = \{h \in \mathcal{C}_n \mid \overline{V_1}^n(h) = 1\} \ .$$

Remark 3.11. From (2.13), the maps $E_{\mathcal{B},N}$ induce injective maps of $\mathbb{K}\mathbf{lein}_n^\infty$ into $\mathbb{K}\mathbf{lein}_{n+1}^\infty$, which will appear to be totally geodesic isometric immersions.

Recall the dimensional constant $r_1(n)$ from (1.4). One can check that

$$r_1(n) = \overline{V_1}^n(1)^{-1} \ ,$$

that is $\overline{V_1}^n(r_1(n)) = 1$, hence $r_1(n) \in \mathbb{K}\mathbf{lein}_n^\infty$ (we also denote by $r_1(n)$ the constant function equal to $r_1(n)$ on $\mathbb{S}^{n-1}$). From the following trivial fact, $\mathbb{K}\mathbf{lein}_n^\infty$ is an open ball in the affine space P_n, in particular it is invariant under convex combinations.

Fact 3.12. *We have* $\mathbb{K}\mathbf{lein}_n^\infty = \{h + r_1(n) | h \in H^1(\mathbb{S}^{n-1})_{01}, -\overline{V_2}^n(h) < \overline{V_2}^n(r_1(n))\}$.

More formally, we have a map

$$\mathrm{Nor}_{\overline{V_1}^n} : \mathbb{H}_n^\infty \to \mathbb{K}\mathbf{lein}_n^\infty \ ,$$

defined by choosing the unique representative of an equivalence class in $\mathbb{H}_n^\infty$ which is in $\mathbb{K}\mathbf{lein}_n^\infty$ (that is, which satisfies $\overline{V_1}^n = 1$). The map $\mathrm{Nor}_{\overline{V_1}^n}$ is an isometry when $\mathbb{K}\mathbf{lein}_n^\infty$ is endowed with the metric

$$d_{\mathbb{K}}(h,k) = d_{\mathbb{H}}([h],[k]) \ .$$

Note that by equation (3.4) we have, for any $h,k \in \mathbb{K}\mathbf{lein}_n^\infty$,

$$d_{\mathbb{K}}(h,k) = d_{\mathcal{H}}(\overline{V_2}^n(h)^{-1/2}h, \overline{V_2}^n(k)^{-1/2}k) = \operatorname{argch}\left(\frac{\overline{V_2}^n(h,k)}{\sqrt{\overline{V_2}^n(h)\overline{V_2}^n(k)}}\right) \ . \tag{3.5}$$

One of the main interest of $\mathbb{K}\mathbf{lein}_n^\infty$ is the following result, which is immediate from Fact 3.1.

Lemma 3.13. *Let $h,k \in \mathbb{K}\mathbf{lein}_n^\infty$. Then the convex combination of h and k is the unique shortest path between h and k for the metric $d_{\mathbb{K}}$.*

We need the following elementary fact.

Fact 3.14. *The function $(\mathcal{H}_n^\infty, d_{\mathcal{H}}) \to \mathbb{R}$, $h \mapsto \int_{\mathbb{S}^{n-1}} h$ is continuous.*

Proof. This function is linear on $H^1(\mathbb{S}^{n-1})_1$, and by Hölder inequality, $|\int_{\mathbb{S}^{n-1}} h| \leq \sqrt{\mathrm{vol}(\mathbb{S}^{n-1})}\|h\|_{L^2} \leq \sqrt{\mathrm{vol}(\mathbb{S}^{n-1})}\|h\|_{H^1}$, hence $\int_{\mathbb{S}^{n-1}} : (\mathcal{H}_n^\infty, \|\cdot\|_{H^1}) \to \mathbb{R}$ is continuous. Then Corollary 3.7 proves the claim. □

Lemma 3.15. *Let $(h_i)_i$ converge to h in $(\mathbb{K}\mathbf{lein}_n^\infty, d_{\mathbb{K}})$. Then $\overline{V_2}^n(h - h_i) \to 0$.*

Proof. By (3.5),

$$\overline{V_2}^n(h_i)^{-1/2}h_i \to \overline{V_2}^n(h)^{-1/2}h$$

in $(\mathcal{H}_n^\infty, d_{\mathcal{H}})$. As $\int_{\mathbb{S}^{n-1}} h_i = \int_{\mathbb{S}^{n-1}} h$ on $\mathbb{K}\mathbf{lein}_n^\infty$, by Fact 3.14 we obtain

$$\overline{V_2}^n(h_i)^{-1/2} \to \overline{V_2}^n(h)^{-1/2} \ .$$

Since $(\mathcal{H}_n^\infty, d_\mathcal{H})$ is locally bi-Lipschitz to $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$, we have

$$\mathbf{p}_\mathcal{H}(\overline{V_2}^n(h_i)^{-1/2}h_i) \to \mathbf{p}_\mathcal{H}(\overline{V_2}^n(h)^{-1/2}h)$$

in $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$, that is $\overline{V_2}^n(h_i)^{-1/2}(h_i - \mathbf{mean}(h_i)) \to \overline{V_2}^n(h)^{-1/2}(h - \mathbf{mean}(h))$ in $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$. Since $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$ is a normed vector space and $\mathbf{mean}(h_i) = \mathbf{mean}(h)$, we obtain $h_i \to h$ in $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$, that is $\sqrt{-\overline{V_2}^n(h-h_i)} \to 0$. □

We use this Lemma, as well as the equivalence of topologies on $\mathcal{H}_n^\infty$ (as stated in Corollary 3.7) to prove the following. We denote by d_{H^1} the distance given by $\|\cdot\|_{H^1}$ on $H^1(\mathbb{S}^{n-1})_1$.

Proposition 3.16. *On $\mathbb{K}\mathbf{lein}_n^\infty$, $d_\mathbb{K}$ and d_{H^1} define the same topology.*

Proof. If $d_\mathbb{K}(h_i, h) \to 0$, then by Lemma 3.15 we have $\overline{V_2}^n(h-h_i) \to 0$, hence since $h - h_i \in H^1(\mathbb{S}^{n-1})_{01}$ we have by Lemma 2.2 $h_i \to h$ for $\|\cdot\|_{H^1}$. Now suppose that $h_i \to h$ for $\|\cdot\|_{H^1}$. By (3.5) and Corollary 3.7, to prove $d_\mathbb{K}(h_i,h) \to 0$ this is sufficient to prove that $\frac{1}{\sqrt{\overline{V_2}^n(h_i)}}h_i \to \frac{1}{\sqrt{\overline{V_2}^n(h)}}h$ for $\|\cdot\|_{H^1}$. Since $h_i \to h$ for $\|\cdot\|_{H^1}$, we have $\overline{V_2}^n(h_i) \to \overline{V_2}^n(h) > 0$, and then the right-hand side of the following inequality goes to zero:

$$\|\frac{1}{\sqrt{\overline{V_2}^n(h_i)}}h_i - \frac{1}{\sqrt{\overline{V_2}^n(h)}}h\|_{H^1} \leq \frac{1}{\sqrt{\overline{V_2}^n(h_i)}}\|h_i - h\|_{H^1} + \left|\frac{1}{\sqrt{\overline{V_2}^n(h_i)}} - \frac{1}{\sqrt{\overline{V_2}^n(h)}}\right| \|h\|_{H^1}.$$

□

Concerning the intersection of the isotropic cone of $\overline{V_2}^n$ with P_n, we have the following property.

Proposition 3.17. *Let $h_i \in \mathbb{K}\mathbf{lein}_n^\infty$ and $k \in \mathbb{K}\mathbf{lein}_n^\infty$. Then*

$$\overline{V_2}^n(h_i) \to 0 \iff d_\mathbb{K}(h_i,k) \to +\infty\ .$$

Proof. First, let us remark that $\mathbf{mean}(h_i) = \mathbf{mean}(k) = r_1(n)$, hence equation (2.5) gives

$$\overline{V_2}^n(h_i,k) = \overline{V_2}^n(h_i - r_1(n), k - r_1(n)) + \overline{V_2}^n(r_1(n)) \tag{3.6}$$

and

$$\overline{V_2}^n(h_i) = \overline{V_2}^n(h_i - r_1(n)) + \overline{V_2}^n(r_1(n)) \text{ and } \overline{V_2}^n(k) = \overline{V_2}^n(k - r_1(n)) + \overline{V_2}^n(r_1(n))\ . \tag{3.7}$$

Assume that $d_\mathbb{K}(h_i,k) = \operatorname{argch}\left(\frac{\overline{V_2}^n(h_i,k)}{\sqrt{\overline{V_2}^n(h_i)\overline{V_2}^n(k)}}\right) \to +\infty$. Since (3.6) implies $\overline{V_2}^n(h_i,k) \leq \overline{V_2}^n(r_1(n))$, we have $\frac{\overline{V_2}^n(h_i,k)}{\sqrt{\overline{V_2}^n(h_i)\overline{V_2}^n(k)}} \leq \frac{\overline{V_2}^n(r_1(n))}{\sqrt{\overline{V_2}^n(h_i)\overline{V_2}^n(k)}}$, hence $\overline{V_2}^n(h_i) \to 0$.

Now assume that $\overline{V_2}^n(h_i) \to 0$. To prove that $d_\mathbb{K}(h_i,k) \to +\infty$ it is sufficient to show that for i large enough, $\overline{V_2}^n(h_i,k)$ is bounded from below by a positive constant. By (3.6) and Cauchy–Schwarz inequality in $H^1(\mathbb{S}^{n-1})_{01}$ we have

$$\overline{V_2}^n(h_i,k) \geq -\sqrt{\overline{V_2}^n(h_i - r_1(n))\overline{V_2}^n(k - r_1(n))} + \overline{V_2}^n(r_1(n))\ .$$

By equation (3.7) we have $\overline{V_2}^n(h_i - r_1(n)) = \overline{V_2}^n(h_i) - \overline{V_2}^n(r_1(n)) \to -\overline{V_2}^n(r_1(n))$, hence we obtain

$$-\sqrt{\overline{V_2}^n(h_i - r_1(n))\overline{V_2}^n(k - r_1(n))} + \overline{V_2}^n(r_1(n)) \to -\sqrt{-\overline{V_2}^n(r_1(n))\overline{V_2}^n(k - r_1(n))} + \overline{V_2}^n(r_1(n))\ .$$

By equation (3.7) we have $-\overline{V_2}^n(k - r_1(n)) < \overline{V_2}^n(r_1(n))$, hence this limit is positive, and this proves that for i large enough, $\overline{V_2}^n(h_i,k)$ is bounded from below by a positive constant. □

4 Spaces of convex bodies

In this section, we are considering spaces of convex bodies in $\mathbb{R}^n$. Convex bodies are determined by their support functions, that are functions living in the spaces introduced in the preceding sections. The intrinsic area V_2 can be written with the help of the bilinear form $\overline{V_2}^n$.

4.1 Support functions

The *support function* $\mathrm{Supp}(K)$ of a convex body K in $\mathbb{R}^n$ gives, at the point $x \in \mathbb{S}^{n-1}$, the distance from the origin of $\mathbb{R}^n$ to the support hyperplane of K with outward normal x. More precisely, $\mathrm{Supp}(K) : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined as

$$\mathrm{Supp}(K)(x) = \max_{p \in K} \langle x, p \rangle \ ,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of $\mathbb{R}^n$.

The support functions are characterized among functions on $\mathbb{S}^{n-1}$ by the following fact, see [32].

Fact 4.1. *A function $h : \mathbb{S}^{n-1} \to \mathbb{R}$ is the support function of a convex body in $\mathbb{R}^n$ if and only if its one homogeneous extension $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$, $\tilde{h}(x) = \|x\| h(x/\|x\|)$, is a convex function.*

Let us recall some elementary facts about the support functions. For the next one, see [18, (2.2.3)]. Recall that we denote by $\mathcal{K}^n$ the space of convex bodies in $\mathbb{R}^n$.

Lemma 4.2. *Let $K \in \mathcal{K}^n$. If $\mathrm{Supp}(K) \leq R$, then $\mathrm{Supp}(K)$ is R-Lipschitz.*

In particular, $\mathrm{Supp}(K)$ belongs to $H^1(\mathbb{S}^{n-1})$, so in turn the support function defines a map

$$\mathrm{Supp} : \mathcal{K}^n \to H^1(\mathbb{S}^{n-1}) \ .$$

Fact 4.3. *The map* Supp *satisfies the following properties:*

- $\mathrm{Supp}(K_1 + K_2) = \mathrm{Supp}(K_1) + \mathrm{Supp}(K_2)$*;*
- $\mathrm{Supp}(\lambda K) = \lambda \,\mathrm{Supp}(K)$*,* $\lambda > 0$*;*
- Supp *is a bijection onto its image;*
- *if* $K_1 \subset K_2$*, then* $\mathrm{Supp}(K_1) \leq \mathrm{Supp}(K_2)$*;*
- $\mathrm{Supp}(\mathcal{K}^n)$ *is a convex cone in* $H^1(\mathbb{S}^{n-1})$*.*

The last point follows because the convex combination of convex bodies is a convex body.

Remark 4.4. Let us warn the reader that if $\mathrm{Supp}(\lambda K) = \lambda \,\mathrm{Supp}(K)$, $\lambda > 0$, we don't have $\mathrm{Supp}(-K) = -\mathrm{Supp}(K)$ in general, where $-K = \{-x | x \in K\}$. Indeed, both $\mathrm{Supp}(-K)$ and $\mathrm{Supp}(K)$ are positive if the origin of $\mathbb{R}^n$ is in the interior of K. Actually, $\mathrm{Supp}(-K)(v) = \mathrm{Supp}(K)(-v)$, and $-\mathrm{Supp}(K)$ is like the support function of K but with the support planes defined by their *inward* unit normals.

The two next lemmas show that support functions of convex bodies have strong convergence properties. For this lemma, see [32, Theorem 1.8.15].

Lemma 4.5. *Let $K_i, K \in \mathcal{K}^n$. If $(\mathrm{Supp}(K_i))_i$ converges pointwise to $\mathrm{Supp}(K)$, then the convergence is uniform.*

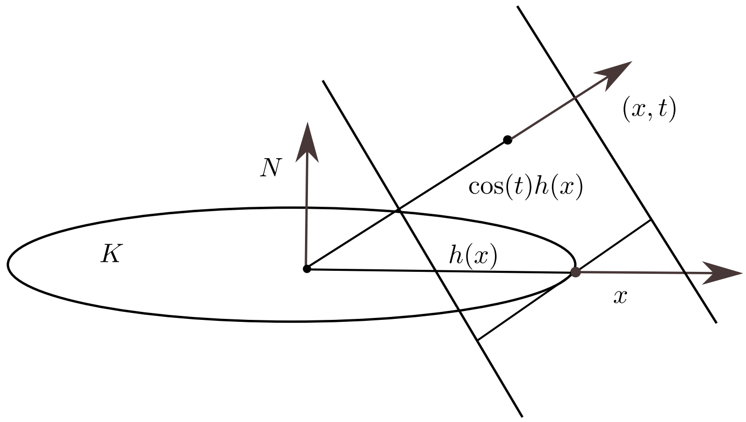

Figure 3: The extension of a support function to a higher dimensional space, see Fact 4.10. Here $h(x) = \mathrm{Supp}(K)(x)$.

For the next result, see [18, Lemma 2.3.4] or [19].

Lemma 4.6. *Let $K_i, K \in \mathcal{K}^n$ such that $(\mathrm{Supp}(K_i))_i$ converges to $\mathrm{Supp}(K)$. Then, almost everywhere, $(\nabla \mathrm{Supp}(K_i))_i$ converges pointwise to $\nabla \mathrm{Supp}(K)$.*

In the following, $C^0(\mathbb{S}^{n-1})$ is the set of continuous functions on $\mathbb{S}^{n-1}$, and for $h \in C^0(\mathbb{S}^{n-1})$ we set $\|h\|_\infty = \sup\{|h(x)|, x \in \mathbb{S}^{n-1}\}$. Let d_∞ be the distance induced by $\|\cdot\|_\infty$ on $\mathrm{Supp}(\mathcal{K}^n)$. Abusing notation, we also denote by Supp the map from $\mathcal{K}^n$ to $C^0(\mathbb{S}^{n-1})$ (instead of Sobolev space) which associates to a convex body its support function.

Remark 4.7. The pull-back of d_∞ onto $\mathcal{K}^n$ is the classical Hausdorff distance, [32, Lemma 1.8.14]. Recall that the Hausdorff distance between K_1 and K_2 in $\mathbb{R}^n$ is the min of the non-negative λ such that $K_1 \subset K_2 + \lambda B^n$ and $K_2 \subset K_1 + \lambda B^n$.

Lemma 4.6 gives the following

Corollary 4.8. *In $\mathrm{Supp}(\mathcal{K}^n)$, if $d_\infty(h_i, h) \to 0$ then $d_{H^1}(h_i, h) \to 0$.*

Proof. This is obvious that $h_i \to h$ in L^2. Moreover, let $R > 0$ be such that $h_i \leq R$ for every i. Then $(\nabla h_i)_i$ almost everywhere converges pointwise to ∇h, hence the convergence holds in L^2 via Lebesgue's dominated convergence: these functions are uniformly bounded by R by Lemma 4.2. Hence $h_i \to h$ for $\|\cdot\|_{H^1}$. □

By definition, $\overline{V_2}^n(\cdot,\cdot)$ is continuous in each entry on $H^1(\mathbb{S}^{n-1})$, hence Corollary 4.8 gives the following.

Corollary 4.9. *$\overline{V_2}^n(\cdot,\cdot)$ is continuous in each entry on $(\mathrm{Supp}(\mathcal{K}^n), d_\infty)$.*

Let us end this section giving a geometric interpretation of the functions $E_{\mathcal{B},N}(h)$ from Section 2.4, see also Figure 3.

Fact 4.10. *Let $K \in \mathcal{K}^n$ and $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a linear isometric embedding. If $\mathcal{B}$ be is the image by ι of the canonical basis of $\mathbb{R}^n$, and N is such that $(\mathcal{B}, N)$ is a direct orthonormal basis of $\mathbb{R}^{n+1}$, then*

$$\mathrm{Supp}(\iota(K)) = E_{\mathcal{B},N}(\mathrm{Supp}(K)) \ .$$

Proof. Denote by $h' \in H^1(\mathbb{S}^n)$ the support function of $\iota(K)$, and let $R_{\mathcal{B},N} \in SO(n+1)$ be the unique rotation sending the canonical basis of $\mathbb{R}^{n+1}$ on $(\mathcal{B}, N)$: we have $R_{\mathcal{B},N}(x,0) = \iota(x)$ for $x \in \mathbb{R}^n$, so in particular $R_{\mathcal{B},N}(p,0) = \iota(c)$ for any $p \in K$. So for any $(x,t) \in \mathbb{S}^{n-1} \times]-\frac{\pi}{2}, \frac{\pi}{2}[$ we have

$$\begin{aligned} h'(\Phi_{\mathcal{B},N}(x,t)) &= \max_{p\in K} \langle \Phi_{\mathcal{B},N}(x,t), \iota(p)\rangle \\ &= \max_{p\in K} \langle R_{\mathcal{B},N}(\cos(t)x, \sin(t)), R_{\mathcal{B},N}(p,0)\rangle \\ &= \max_{p\in K} \langle \cos(t)x, p\rangle = \cos(t)h(x)\ . \end{aligned}$$

□

4.2 Properties of the intrinsic area

A $C^2_+(n)$ convex body K is a convex body in $\mathbb{R}^n$ with C^2 boundary and such that its Gauss map is a diffeomorphism from the boundary of K onto $\mathbb{S}^{n-1}$. We will repeatedly use the following approximation result, see e.g. [18, Lemma 2.3.3].

Fact 4.11. $\mathrm{Supp}(C^2_+(n))$ *is dense in* $(\mathrm{Supp}(\mathcal{K}^n), d_\infty)$.

If $K \in C^2_+(n)$, then the eigenvalues of the second differential of $\widetilde{\mathrm{Supp}(K)}$ (the one-homogeneous extension of $\mathrm{Supp}(K)$) at $x \in \mathbb{S}^{n-1}$ are 0 (with eigenvector x) and positive numbers $r_1, \dots, r_{n-1}$, which are the principal radii of curvature of (the boundary of) K. We can consider the r_i as functions on $\mathbb{S}^{n-1}$ [32, Corollary 2.5.2]. A change of variable provided by the Gauss map gives [32, p. 58, 119]

$$\mathrm{vol}_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K) r_1 \cdots r_{n-1}\ .$$

Now for $\epsilon > 0$, as on the one hand the Gauss map of B^n is the identity, and on the other hand if λ is an eigenvalue of a matrix A, then $\lambda + \epsilon$ is an eigenvalue of $A + \epsilon\,\mathrm{Id}$, we obtain

$$\mathrm{vol}_n(K + \epsilon B^n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} (\mathrm{Supp}(K) + \epsilon)(r_1 + \epsilon)\cdots(r_{n-1} + \epsilon)\ , \tag{4.1}$$

and Steiner formula (1.1) follows for $C^2_+(n)$ convex bodies. Note that we can make a polynomial interpolation for the intrinsic volumes: given $n+1$ different positive numbers ϵ, (1.1) can be considered as a solvable linear system of $d+1$ equations with $V_i(K)$ as unknowns —the matrix of the system is a Vandermonde matrix. In turn, the $V_i(K)$ are linear combination of volumes, hence the maps $V_i : \mathcal{K}^n \to \mathbb{R}$ are continuous for the Hausdorff topology as the volume is continuous for the Hausdorff topology on the space of convex bodies [32].

From (4.1) we obtain in particular

$$V_1(K) = \frac{1}{n\kappa_{n-1}}\int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K) + r_1 + \cdots + r_{n-1}\ .$$

But $r_1 + \cdots + r_{n-1}$ is the trace of the second differential of $\widetilde{\mathrm{Supp}(K)}$, and since for 1-homogeneous functions f on $\mathbb{R}^n$ we have, as functions on $\mathbb{S}^{n-1}$,

$$\Delta_e f = (n-1)f + \Delta f\ ,$$

where Δ_e is the Laplacian of $\mathbb{R}^n$ and Δ is the Laplacian of $\mathbb{S}^{n-1}$, we have

$$r_1 + \cdots + r_{n-1} = (n-1)\,\mathrm{Supp}(K) + \Delta\,\mathrm{Supp}(K)\ . \tag{4.2}$$

As by Green formula $\int_{\mathbb{S}^{n-1}} \Delta\,\mathrm{Supp}(K) = 0$ we obtain

$$V_1(K) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K)\ .$$

Note that this gives

$$V_1(K) = \overline{V_1}^n(\mathrm{Supp}(K))\ .$$

Clearly, by Fact 4.11, the formula above holds for any convex body.

Remark 4.12. By Fact 4.10 and (2.13), we obtain that if $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a linear isometric embedding, then

$$V_1(\iota(K)) = V_1(K)\ .$$

Since $\overline{V_1}^n(\mathrm{Supp}(K)) = \frac{n\kappa_n}{\kappa_{n-1}}\mathbf{mean}(\mathrm{Supp}(K))$, the quantity $V_1(K)$ is intimately related to the *mean width* of K, which is equal to $2\mathbf{mean}(\mathrm{Supp}(K))$.

We will use a Minkowski type formula, see Proposition 4.3.1 as well as the formula in the bottom of p. 45 and Remark 4.3.2 in [30]:

$$\int_{\mathbb{S}^{n-1}} \sum_{i\neq j} r_i r_j = \frac{n-2}{2} \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K)(r_1 + \cdots + r_{n-1})\ .$$

On the other hand, we also obtain from (4.1) that

$$V_2(K) = \frac{1}{n\kappa_{n-2}} \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K)(r_1 + \cdots + r_{n-1}) + \sum_{i\neq j} r_i r_j$$

so

$$V_2(K) = \frac{1}{2\kappa_{n-2}} \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K)(r_1 + \cdots + r_{n-1}) \tag{4.3}$$

$$\overset{(2.11),(4.2)}{=} c_n \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K)(\mathrm{Supp}(K) + (n-1)^{-1}\Delta\,\mathrm{Supp}(K))$$

and using Green formula,

$$V_2(K) = \overline{V_2}^n(\mathrm{Supp}(K))\ .$$

By density and continuity on both sides of the equation above, the equality above holds actually for any convex body, that gives the bridge with the preceding sections. Seemly this formula first appeared in [19, Theorem 4.2a], see also [32, p. 298] or [18, Proposition 2.4.2].

Let us check the properties of the intrinsic area stated in the introduction. Most of them need to be checked for the $C^2_+(n)$ case, and then use again an approximation argument. For any $p \in \mathbb{R}^n$, $V_2(K + \{p\}) = V_2(K)$, so up to a translation we can consider that the origin is in the relative interior of K, that gives $\mathrm{Supp}(K) \geq 0$. As in the $C^2_+(n)$ case, $r_i > 0$, A2) is immediate from (4.3). A6) follows immediately from Proposition 2.11 and Fact 4.10.

A3) and A4) will be obtained as a byproduct of properties of the mixed-area: A3) is a consequence of M5), and A4) is a consequence of M7).

From (1.5) and Green formula, one has

$$V_2(K_1, K_2) = \overline{V_2}^n(\mathrm{Supp}(K_1), \mathrm{Supp}(K_2))\ ,$$

and this gives properties M3) and M4) of the mixed-area. Moreover in the $C^2_+(n)$ case,

$$V_2(K_1, K_2) = c_n \int_{\mathbb{S}^{n-1}} \mathrm{Supp}(K_1)(r_1(K_2) + \cdots + r_{n-1}(K_2))\ ,$$

and this gives property M5). If K is a point, it is immediate from (1.5) that $V_2(K,Q)=0$ for any Q, that gives an implication of M6). The first part of M7) follows from that and from M5), by taking a point for K_1. The second part of M7) follows from Fact 2.4 and the following observation: if Q is a segment, for any other segment K with a different direction, $Q+K$ is a parallelogram with non-empty interior, and (1.5) gives that $V_2(K,Q)>0$. The remaining implication in M6) follows from M7).

The *Steiner point* of a convex body is the following point of $\mathbb{R}^n$:

$$\mathbf{stein}(K)=\frac{1}{\kappa_n}\int_{\mathbb{S}^{n-1}}\mathrm{Supp}(K)(x)x\mathrm{d}\mathbb{S}^{n-1}\ .$$

We will denote by $\mathcal{K}^n_S$ the space of convex bodies in $\mathbb{R}^n$ with the origin as Steiner point, and by $\mathcal{K}^{n*}_S$ the subset of convex bodies with positive intrinsic area. For clarity, let us note the following obvious fact.

Fact 4.13. *For $K\in\mathcal{K}^n$ we have*

$$\mathbf{stein}(K)=0\Longleftrightarrow\mathrm{Supp}(K)\in H^1(\mathbb{S}^{n-1})_1\ ,$$

so in particular

$$\mathrm{Supp}(\mathcal{K}^n_S)\subset H^1(\mathbb{S}^{n-1})_1\ .$$

Moreover, if $K\in\mathcal{K}^n_S$ is not a point then $\mathrm{Supp}(K)\in\overline{\mathcal{C}}_n$ *(see Section 2.3), and if $K\in\mathcal{K}^n_S$ is not a point or a segment we have* $\mathrm{Supp}(K)\in\mathcal{C}_n$*, that is*

$$\mathrm{Supp}(\mathcal{K}^{n*}_S)\subset\mathcal{C}_n\ .$$

Let K_1,K_2 be two convex bodies. M8) does not change if a translation is performed on K_1 or K_2, so we can consider that the Steiner point of both K_1 and K_2 is the origin. If K_1 or K_2 are points then M8) is obvious. Otherwise, $\mathrm{Supp}(K_1)$ and $\mathrm{Supp}(K_2)$ belong to $\overline{\mathcal{C}}_n$, so Fact 2.7 (reversed Cauchy–Schwarz inequality) gives M8) (Alexandrov–Fenchel inequality). Note that this result corresponds to classical geometric inequalities in $\mathbb{R}^3$ (see [32, p.387], [24]). In $\mathbb{R}^2$, it is the famous Minkowski inequality, and as we have seen in the introduction, if one of the two convex bodies is a disc, this gives the isoperimetric inequality.

Remark 4.14. Note that M6) is related to Fact 2.1. Indeed, support functions of points are exactly the restriction of linear forms to the sphere: if $p\in\mathbb{R}^n$, then $\mathrm{Supp}(\{p\})=\langle p,\cdot\rangle$. And these functions are exactly the kernel of $\overline{V_2}^n$ on $H^1(\mathbb{S}^{n-1})$.

4.3 Comparison of topologies

Recall that d_∞ is the distance induced by $\|h\|_\infty=\sup\{|h(x)|,x\in\mathbb{S}^{n-1}\}$. One of the principal features of d_∞ is the classical Blaschke selection theorem, which says that from each bounded sequence of convex bodies in $\mathbb{R}^n$ one can extract a converging subsequence for the Hausdorff distance. Actually it is a direct consequence of Lemma 4.2 and the Arzela–Ascoli Theorem. Let us state it in the following way.

Theorem 4.15 (Blaschke selection theorem)**.** $(\mathrm{Supp}(\mathcal{K}^n),d_\infty)$ *is a proper metric space.*

Let $\mathcal{K}^n_{SV_1}$ be the subset of $\mathcal{K}^n_S$ (convex bodies with Steiner point at the origin) of convex bodies K such that $V_1(K)=1$. Elements of $\mathcal{K}^n_{SV_1}$ are sometimes called *normalized convex bodies*, see p. 164 and after in [32]. Note that $\mathcal{K}^n_{SV_1}$ is stable under convex combinations of convex bodies.

Fact 4.16. *If $K\in\mathcal{K}^n_{SV_1}$, then K is included in the closed unit ball in $\mathbb{R}^n$.*

Proof. If this is not true, then there exists $x \in K$ with $\|x\| > 1$. The Steiner point of K is 0 and belongs to K, hence $[0,x] \subset K$, which gives $V_1(K) \geq V_1([0,x]) = \|x\| > 1$: this is a contradiction. □

Then Blaschke selection theorem (Theorem 4.15) implies the following

Corollary 4.17. $(\mathrm{Supp}(\mathcal{K}^n_{SV_1}), d_\infty)$ *is compact.*

Proof. This is sufficient to show that $\mathrm{Supp}(\mathcal{K}^n_{SV_1})$ is closed and bounded in $(\mathrm{Supp}(\mathcal{K}^n), d_\infty)$. The boundedness is a direct consequence of Fact 4.16: for every $h \in \mathrm{Supp}(\mathcal{K}^n_{SV_1})$ we have $\|h\|_\infty \leq 1$. And closeness is clear, since the maps $h \mapsto \int_{\mathbb{S}^{n-1}} h(x) x \mathrm{d}\mathbb{S}^{n-1}(x)$ and $h \mapsto \int_{\mathbb{S}^{n-1}} h(x) \mathrm{d}\mathbb{S}^{n-1}(x)$ are continuous on $(\mathrm{Supp}(\mathcal{K}^n), d_\infty)$. □

Let $\mathcal{K}^{n*}_{SV_1}$ the subset of $\mathcal{K}^n_{SV_1}$ of convex bodies of positive intrinsic area: it is clear that $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$ is a convex subset of $\mathbb{K}\mathbf{lein}^\infty_n$. The goal is to prove that the analog of Theorem 4.15 holds on $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\mathbb{K})$. As a tool, we will use the distance d_{L^2} induced by the L^2 norm on $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, as well as the following theorem, see [35] and [18, Proposition 2.3.1].

Theorem 4.18 (Vitale)**.** *The distances d_∞ and d_{L^2} induce the same topology on* $\mathrm{Supp}(\mathcal{K}^n) \subset C^0(\mathbb{S}^{n-1})$.

The result is weaker than saying that the two norms are equivalent on the space of convex bodies, that is not true, see [35] for details.

Corollary 4.19. *The distances d_∞, d_{L^2} and d_{H^1} induce the same topology on* $\mathrm{Supp}(\mathcal{K}^n)$.

Proof. We prove that d_{L^2} and d_{H^1} induce the same topology. If $h_i \to h$ for $\|\cdot\|_{H^1}$ then obviously $h_i \to h$ for $\|\cdot\|_{L^2}$. And if $h_i \to h$ for $\|\cdot\|_{L^2}$ then by Theorem 4.18 we have $h_i \to h$ for d_∞, and Corollary 4.8 proves the claim. □

A direct consequence of Proposition 3.16 and Corollary 4.19 is the following corollary, which relate the distances d_∞ and $d_\mathbb{K}$.

Corollary 4.20. *On* $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$*, d_∞ and $d_\mathbb{K}$ (as well as d_{L^2} and d_{H^1}) induce the same topology.*

Proposition 4.21. $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\mathbb{K})$ *is a proper metric space.*

Proof. Let A be a closed bounded subset of $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\mathbb{K})$. We want to show that A is compact for $d_\mathbb{K}$; by Corollary 4.20, it suffices to show that it is compact for d_∞. And by Corollary 4.17, it suffices to show that A is closed in $(\mathrm{Supp}(\mathcal{K}^n_{SV_1}), d_\infty)$.

So assume $(h_i)_i$ is a sequence of elements of A converging to $h \in \mathrm{Supp}(\mathcal{K}^n_{SV_1})$ for d_∞; we want to show that $h \in A$. If $h \in \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, then this is true, because Corollary 4.20 implies that A is a closed subset of $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\infty)$. Otherwise, $h \in \mathrm{Supp}(\mathcal{K}^n_{SV_1}) \setminus \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, hence $\overline{V_2}^n(h) = 0$ and by Corollary 4.9 we have $\overline{V_2}^n(h_i) \to 0$. Then by Proposition 3.17, the distance in $(\mathbb{K}\mathbf{lein}^\infty_n, d_\mathbb{K})$ between h_i and any given point $k \in \mathbb{K}\mathbf{lein}^\infty_n$ goes to infinity, and that contradicts the fact that A is a bounded subset of $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_\mathbb{K})$. □

Remark 4.22. Let us give an illustration of the fact that, even if d_∞ and $d_\mathbb{K}$ induce the same topology, their behavior is quite different. First, recall that segments are shortest paths for the Hausdorff metric. Indeed, for any $K, L \in \mathcal{K}^n$ and $t \in (0,1)$, $d_\infty(\mathrm{Supp}(K), (1-t)\,\mathrm{Supp}(K) + t\,\mathrm{Supp}(L)) = \|\mathrm{Supp}(K) - (1-t)\,\mathrm{Supp}(K) - t\,\mathrm{Supp}(L)\|_\infty = t\|\mathrm{Supp}(K) - \mathrm{Supp}(L)\|_\infty = t d_\infty(K, L)$. But conversely to $d_\mathbb{K}$, in general, segments are not the only shortest path between two given convex bodies, see note 11 of Section 1.8 in [32].

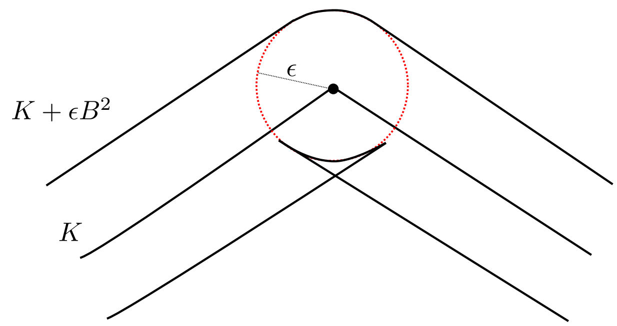

Figure 4: If a plane convex body K has a non-smooth point, then for any $\epsilon > 0$, $\mathrm{Supp}(K) + \epsilon\,\mathrm{Supp}(B^2)$ is the support function of a convex body, while $\mathrm{Supp}(K) - \epsilon\,\mathrm{Supp}(B^2)$ is not.

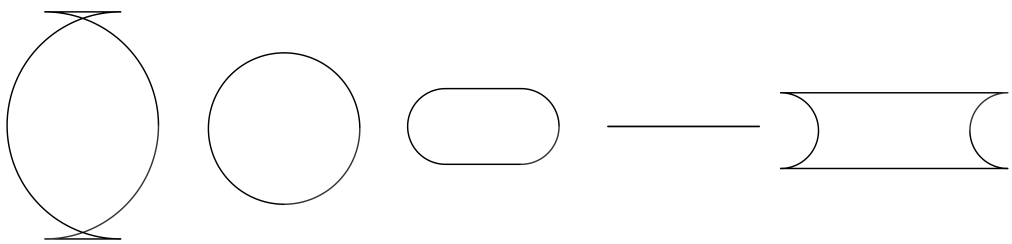

Figure 5: The disc and the segment $[-1, 1]$ are both terminal points of the segment joining them.

4.4 Terminal points of segments

Let $K_1, K_2 \in \mathcal{K}^n_{SV_1}$. The *segment* between K_1 and K_2 is $\{(1-t)K_1 + tK_2, t \in [0,1]\}$. We say that $K_1 \in \mathcal{K}^n_{SV_1}$ is a *terminal point* of the segment if for any $t < 0$, $(1-t)\,\mathrm{Supp}(K_1) + t\,\mathrm{Supp}(K_2) \notin \mathrm{Supp}(\mathcal{K}^n_{SV_1})$. An *extreme point* K of $\mathcal{K}^n_{SV_1}$ is such that there does not exist $K_1, K_2 \in \mathcal{K}^n_{SV_1}$, $K_1 \neq K_2$, and $t \in (0,1)$ such that $\mathrm{Supp}(K) = (1-t)\,\mathrm{Supp}(K_1) + t\,\mathrm{Supp}(K_2)$. In the plane, extreme points of $\mathcal{K}^2_{SV_1}$ are segments and triangles [32, Theorem 3.2.14]. For $n \geq 3$, extreme points of $\mathcal{K}^n_{SV_1}$ are dense for the Hausdorff metric [32, 3.2.18].

Clearly, an extreme point is a terminal point for all the segments ending at this point. But there are much more terminal points. For example, one can find convex bodies with a non smooth point on the boundary (i.e. a point of the convex body contained in more than one support plane) which are terminal points for the segment starting at the unit ball —this idea is illustrated in Figure 4.

In this section, we will use a different argument to prove that *any* convex body is the terminal point of some segment (Proposition 4.24). The simplest case illustrating our argument is depicted in Figure 5.

By Fact 4.1, if a function $h \in \mathbb{K}\mathbf{lein}^\infty_n$ belongs to $\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, then the Laplacian of $\tilde{h}$ is non-negative, in the weak sense ($\tilde{h}$ is the one-homogenous extention of h). Let $C^\infty_c(\mathbb{R}^n)$ be the set of smooth functions with compact support in $\mathbb{R}^n$: this means that for every non-negative function $\varphi \in C^\infty_c(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \tilde{h}(x)\Delta_e\varphi(x)\mathrm{d}x \geq 0 \ .$$

For $1 \leq p < n$ we will denote by $B_{p,n}$ the p-dimensional ball with radius $r_1(p)$ in $\mathbb{R}^n$, which is the set of points $x \in \mathbb{R}^n$ with $x_1^2 + \cdots + x_p^2 \leq r_1(p)^2$ and $x_{p+1} = \cdots = x_n = 0$. We have $V_1(B_{p,n}) = 1$, hence $B_{p,n} \in \mathcal{K}^n_{SV_1}$ (note that $B_{p,n} \in \mathcal{K}^{n*}_{SV_1}$ if and only if $p \geq 2$). Let $b_{p,n} = \mathrm{Supp}(B_{p,n}) \in \mathrm{Supp}(\mathcal{K}^n_{SV_1})$ and let $\widetilde{b_{p,n}}(x) = r_1(p)\sqrt{x_1^2 + \cdots + x_p^2}$ be the 1-homogeneous extension of $b_{p,n}$ (if $p = 1$, then $\widetilde{b_p}(x) = r_1(1)|x_1| = \frac{|x_1|}{2}$).

Example 4.23 (The $n-$dimensional ball is the terminal point of a segment)**.** Assume $n \geq 3$, and let $p \in \mathbb{N}$ be such that $2 \leq p < n$. Let K be the $n-$dimensional ball with radius $r_1(n)$ in $\mathbb{R}^n$: we have $V_1(K) = 1$, hence $K \in \mathcal{K}^n_{SV_1}$, and let $k = \mathrm{Supp}(K) \in \mathrm{Supp}(\mathcal{K}^n_{SV_1})$ be its support function. Then k is the terminal point of the segment starting at $b_{p,n}$.

Indeed, the 1-homogeneous extension of the support function of K is

$$\tilde{k}(x) = r_1(n)\sqrt{x_1^2 + \cdots + x_n^2} \ ,$$

and $\Delta_e \tilde{k}(x) = \frac{r_1(n)(n-1)}{\sqrt{x_1^2+\cdots+x_n^2}}$. Also $\Delta_e \widetilde{b_{p,n}}(x) = \frac{r_1(p)(p-1)}{\sqrt{x_1^2+\cdots+x_p^2}}$. Now let $\epsilon > 0$ and consider the point $x_\epsilon \in \mathbb{R}^n$ such that $x_1 = \cdots = x_p = \epsilon$ and $x_{p+1} = \cdots = x_n = 1$. For $t < 0$ we have

$$\Delta_e((1-t)\tilde{k} + t\widetilde{b_{p,n}})(x_\epsilon) = \frac{(1-t)r_1(n)(n-1)}{\sqrt{p\epsilon^2 + (n-p)}} + \frac{tr_1(p)(p-1)}{\sqrt{p}\epsilon} \ ,$$

and this quantity goes to $-\infty$ when ϵ goes to zero. This shows that for $t < 0$, $(1-t)k + tb_{p,n} \notin \mathrm{Supp}(\mathcal{K}^n_{SV_1})$.

By using the same arguments we obtain the following.

Proposition 4.24. *Let $p \in \mathbb{N}$ such that $1 \leq p < n$. Then any $K \in \mathcal{K}^{n*}_{SV_1}$ is the terminal point of a segment in $\mathcal{K}^{n*}_{SV_1}$, which starts at some embedded $p-$dimensional ball in $\mathbb{R}^n$.*

In fact the proof shows that there are infinitely many such segments.

Remark 4.25. If $p = 1$, this ball is in fact a segment and lies on the boundary of $\mathbb{K}\mathbf{lein}_n^\infty$.

To prove this Proposition we need the following theorem due to Alexandrov (see [7]):

Theorem 4.26. *A convex function $f : \mathbb{R}^n \to \mathbb{R}$ is* twice differentiable *at almost every $\bar{x} \in \mathbb{R}^n$, which means that for almost every $\bar{x} \in \mathbb{R}^n$ there exists a quadratic polynomial $Q_{\bar{x}}$ and a function $R_{\bar{x}}$ such that*

$$f(x) = Q_{\bar{x}}(x) + R_{\bar{x}}(x) \ \text{and} \ \lim_{u \to 0} \frac{R_{\bar{x}}(\bar{x} + u)}{\|u\|^2} = 0 \ .$$

Proof of Proposition 4.24. Let $k = \mathrm{Supp}(K) \in \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$, and let $\tilde{k}$ be its 1-homogeneous extension. Let $\bar{x} \in \mathbb{R}^n$ be a point at which $\tilde{k}$ is twice differentiable, and let $Q_{\bar{x}}$ and $R_{\bar{x}}$ be as in Alexandrov's theorem. Since $n > p$, the vector space $\{x_1 = \cdots = x_p = 0\}$ has positive dimension, hence up to a rotation of K we may assume that the first components of $\bar{x}$ are $\bar{x}_1 = \cdots = \bar{x}_p = 0$.

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a non-negative function, with support in the unit ball in $\mathbb{R}^n$, positive in a neighborhood of 0, and with $\int_{\mathbb{R}^n} \varphi = 1$. For $\epsilon > 0$, let $\varphi_\epsilon \in C_c^\infty(\mathbb{R}^n)$ be the function $\varphi_\epsilon(x) = \frac{1}{\epsilon^n}\varphi(\frac{x-\bar{x}}{\epsilon})$: this function is non-negative, has support in $B(\epsilon, \bar{x})$ (the ball centered at $\bar{x}$ and with radius ϵ), and $\int_{\mathbb{R}^n} \varphi_\epsilon = 1$.

Let $t < 0$. We want to show that $(1-t)k + tb_{p,n} \notin \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$. We argue by contradiction: assume that $(1-t)k + tb_{p,n} \in \mathrm{Supp}(\mathcal{K}^{n*}_{SV_1})$. Then $(1-t)\tilde{k} + t\widetilde{b_{p,n}}$ is a convex function on $\mathbb{R}^n$, hence its Laplacian is non-negative in the weak sense, so in particular we have

$$\int_{\mathbb{R}^n} ((1-t)\tilde{k} + t\widetilde{b_{p,n}})\Delta_e \varphi_\epsilon \geq 0 \ . \tag{4.4}$$

We will first show that we always have

$$\int_{\mathbb{R}^n} \tilde{k}\Delta_e\varphi_\epsilon \underset{\epsilon\to 0}{\longrightarrow} +\infty \ . \tag{4.5}$$

Since t is negative, with equation (4.4) it is sufficient to show that

$$\int_{\mathbb{R}^n} \widetilde{b_{p,n}}\Delta_e\varphi_\epsilon \underset{\epsilon\to 0}{\longrightarrow} +\infty \ . \tag{4.6}$$

Now we need to argue depending whether $p = 1$ or $p \geq 2$.

- If $p \geq 2$ we have $\Delta_e\widetilde{b_{p,n}}(x) = \frac{r_1(p)(p-1)}{\sqrt{x_1^2+\cdots+x_p^2}}$, and since $\bar{x}_1 = \cdots = \bar{x}_p = 0$ we have $\sqrt{x_1^2+\cdots+x_p^2} \leq \|x-\bar{x}\|$, hence $\Delta_e\widetilde{b_{p,n}}(x) \geq \frac{r_1(p)(p-1)}{\epsilon}$ for every $x \in B(\epsilon,\bar{x})$, so we have (by using Green's formula)

  $$\int_{\mathbb{R}^n} \widetilde{b_{p,n}}\Delta_e\varphi_\epsilon = \int_{B(\epsilon,\bar{x})} \varphi_\epsilon\Delta_e\widetilde{b_{p,n}} \geq \frac{r_1(p)(p-1)}{\epsilon}\int_{B(\epsilon,\bar{x})}\varphi_\epsilon = \frac{r_1(p)(p-1)}{\epsilon} \ ,$$

  and this gives (4.6).

- If $p = 1$, then we have

  $$\begin{aligned}\int_{\mathbb{R}^n} \widetilde{b_{p,n}}(x)\Delta_e\varphi_\epsilon(x)\mathrm{d}x &= \frac{1}{2}\int_{\mathbb{R}^n} |x_1|\Delta_e\varphi_\epsilon(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^{n-1}} \varphi_\epsilon(0,x_2,\ldots,x_n)\mathrm{d}x_2\ldots\mathrm{d}x_n \\ &= \frac{1}{\epsilon^n}\int_{\mathbb{R}^{n-1}} \varphi\left(0,\frac{x_2-\bar{x}_2}{\epsilon},\ldots,\frac{x_n-\bar{x}_n}{\epsilon}\right)\mathrm{d}x_2\ldots\mathrm{d}x_n \\ &= \frac{1}{\epsilon}\int_{\mathbb{R}^{n-1}} \varphi(0,y_2,\ldots,y_n)\mathrm{d}y_2\ldots\mathrm{d}y_n \ :\end{aligned}$$

  the second equality is a classical computation, the third is true because $\bar{x}_1 = 0$, and for the last one we use the change of variable $y_i = \frac{x_i-\bar{x}_i}{\epsilon}$. Since φ is positive in a neighborhood of zero we have $\int_{\mathbb{R}^{n-1}} \varphi(0,y_2,\ldots,y_n)\mathrm{d}y_2\ldots\mathrm{d}y_n > 0$, and this gives (4.6).

Moreover, since $\widetilde{k} = Q_{\bar{x}} + R_{\bar{x}}$ we have

$$\int_{\mathbb{R}^n} \widetilde{k}\Delta_e\varphi_\epsilon = \int_{\mathbb{R}^n} Q_{\bar{x}}\Delta_e\varphi_\epsilon + \int_{\mathbb{R}^n} R_{\bar{x}}\Delta_e\varphi_\epsilon \ .$$

The function $Q_{\bar{x}}$ is a quadratic polynomial, hence its Laplacian is equal to a constant $C \in \mathbb{R}$, which gives $\int_{\mathbb{R}^n} Q_{\bar{x}}\Delta_e\varphi_\epsilon = \int_{\mathbb{R}^n} C\varphi_\epsilon = C$. And since $\Delta_e\varphi_\epsilon(x) = \frac{1}{\epsilon^{n+2}}\Delta_e\varphi(\frac{x-\bar{x}}{\epsilon})$, with the change of variable $y = \frac{x-\bar{x}}{\epsilon}$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} R_{\bar{x}}(x)\Delta_e\varphi_\epsilon(x)\mathrm{d}x &= \frac{1}{\epsilon^{n+2}}\int_{B(\epsilon,\bar{x})} R_{\bar{x}}(x)\Delta_e\varphi\left(\frac{x-\bar{x}}{\epsilon}\right)\mathrm{d}x \\ &= \frac{1}{\epsilon^2}\int_{B(1,0)} R_{\bar{x}}(\bar{x}+\epsilon y)\Delta_e\varphi(y)\mathrm{d}y \ .\end{aligned}$$

Since $\frac{R_{\bar{x}}(\bar{x}+u)}{\|u\|^2} \underset{u\to 0}{\longrightarrow} 0$, there exists $M > 0$ such that $|R_{\bar{x}}(\bar{x}+u)| \leq M\|u\|^2$ for $\|u\|$ small enough, hence for ϵ small enough we have, for every $y \in B(1,0)$, $|R_{\bar{x}}(\bar{x}+\epsilon y)| \leq M\epsilon^2\|y\|^2$, hence we obtain

$$|\int_{\mathbb{R}^n} R_{\bar{x}}(x)\Delta_e\varphi_\epsilon(x)\mathrm{d}x| \leq M\int_{B(1,0)} \|y\|^2|\Delta_e\varphi(y)|\mathrm{d}y \ .$$

The integral $\int_{\mathbb{R}^n} R_{\bar{x}} \Delta_e \varphi_\epsilon$ does not go to $+\infty$ when ϵ goes to zero, and by (4.5) this is a contradiction. $\square$

4.5 Proof of Theorems 1 and 2

Remember that the notations used in this article are described at the end of the introduction. Let $\mathcal{K}^{n*}_{SH}$ be the set of convex bodies in $\mathbb{R}^n$, which are not points nor segments, with Steiner point equal to 0, and modulo positive homotheties. Let d_{SH} be the distance on $\mathcal{K}^{n*}_{SH}$ such that the projection map

$$(\mathcal{K}^{n*}_{SH}, d_{SH}) \to (\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$$

is an isometry. We obviously have the inclusion $\mathrm{Supp}(\mathcal{K}^{n*}_{SH}) \subset \mathbb{H}^{\infty}_n$, and by construction of the metric we have the following:

$$(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n}) \text{ is isometric to } (\mathrm{Supp}(\mathcal{K}^{n*}_{SH}), d_{\mathbb{H}}) \text{ and } (\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}}) \ .$$

Moreover, $\mathrm{Supp}(\mathcal{K}^{n*}_S)$ is a convex cone in $\mathcal{C}_n$. Since $\mathcal{K}^{n*}_{SH}$ is the quotient of $\mathcal{K}^{n*}_S$ by the action of positive homotheties, and $\mathbb{H}^{\infty}_n$ is the projective quotient of $\mathcal{C}_n$, we obtain that $\mathrm{Supp}(\mathcal{K}^{n*}_{SH})$ is a convex subset of $\mathbb{H}^{\infty}_n$.

This gives part of Theorem 2, as well as part of Theorem 1: $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ is isometric to a convex subset of $\mathbb{H}^{\infty}_n$. It is a uniquely geodesic metric space, the unique shortest path between $[K_1]$ and $[K_2]$ being $[(1-t)K_1 + tK_2]$, $t \in [0,1]$, and it has curvature bounded from below and above by -1 in the sense of Alexandrov (see Section 5.1 for the definition of this property). It is a proper metric space by Proposition 4.21, and Proposition 4.24 gives the non-extendability property for shortest paths.

Since $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ is proper, it is complete, hence $(\mathrm{Supp}(\mathcal{K}^{n*}_{SH}), d_{\mathbb{H}})$ is also complete, so $\mathrm{Supp}(\mathcal{K}^{n*}_{SH}) \subset \mathbb{H}^{\infty}_n$ is a closed subspace. Now, let us prove that $\mathrm{Supp}(\mathcal{K}^{n*}_{SH})$ has empty interior. If this is not true, then there exists a ball B in $(\mathbb{H}^{\infty}_n, d_{\mathbb{H}})$ such that $B \subset \mathrm{Supp}(\mathcal{K}^{n*}_{SH})$; we can even assume that $\bar{B}$ (the closure of B) satisfies $\bar{B} \subset \mathrm{Supp}(\mathcal{K}^{n*}_{SH})$. Since $(\mathrm{Supp}(\mathcal{K}^{n*}_{SH}), d_{\mathbb{H}})$ is proper, closed balls are compact, hence $\bar{B}$ is compact. Hence there exists a non-empty relatively compact open set in $(\mathbb{K}\mathbf{lein}^{\infty}_n, d_{\mathbb{K}})$. Hence by Corollary 3.5 this is also true for the infinite-dimensional Banach space $(H^1(\mathbb{S}^{n-1})_{01}, d_{01})$, and that is impossible: a closed ball would be compact.

Moreover $\mathrm{Supp}(\mathcal{K}^{n*}_{SH})$ contains an entire geodesic of $\mathbb{H}^{\infty}_n$: it is sufficient to do it for $n = 2$. In the plane, consider the following segments: $K_1 = [-1,1] \times \{0\}$ and $K_2 = \{0\} \times [-1,1]$. For $0 \le t \le 1$, the convex combination $(1-t)K_1 + tK_2$ is the rectangle $[-(1-t), 1-t] \times [-t, t]$, whose Steiner point is 0. This gives an entire geodesic of $\mathbb{H}^{\infty}_2$ contained in $\mathrm{Supp}(\mathcal{K}^{2*}_{SH})$.

The convex hyperbolic polyhedra constructed in [3] parametrize the shapes of convex polygons with fixed angles; by construction, they isometrically embed into $(\mathscr{O}\mathscr{S}hape^{2*}, d_{\mathscr{O}\mathscr{S}^2})$. As the dimension of the hyperbolic polyhedra is $(s-3)$ if the polygons have s edges, and by the isometric immersions of $(\mathscr{O}\mathscr{S}hape^{2*}, d_{\mathscr{O}\mathscr{S}^2})$ into any $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ induced by any linear isometric immersion of $\mathbb{R}^2$ into $\mathbb{R}^n$, we arrive at the following fact, which in particular shows that $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$ has infinite Hausdorff dimension:

Fact 4.27. *For any $p \in \mathbb{N}$, there is an open ball of the finite dimensional hyperbolic space $\mathbb{H}^p$ that isometrically embeds into $(\mathscr{O}\mathscr{S}\mathrm{hape}^{n*}, d_{\mathscr{O}\mathscr{S}^n})$.*

Let us prove the assertion about the boundary $\partial\mathscr{O}\mathscr{S}hape^{n*}$ of $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$. It is the space of segments, up to translations and homotheties: indeed, for example by looking at the isometric model $(\mathrm{Supp}(\mathcal{K}^{n*}_{SV_1}), d_{\mathbb{K}})$, we see that the convex bodies K on the boundary are the one for which $V_2(K) = 0$ (see Proposition 3.17) and $V_1(K) = 1$, and these are exactly segments. Hence $\partial\mathscr{O}\mathscr{S}hape^{n*}$ is in bijection with $P^{n-1}(\mathbb{R})$, the real projective space of dimension $n-1$ (that is, the space of lines in $\mathbb{R}^n$).

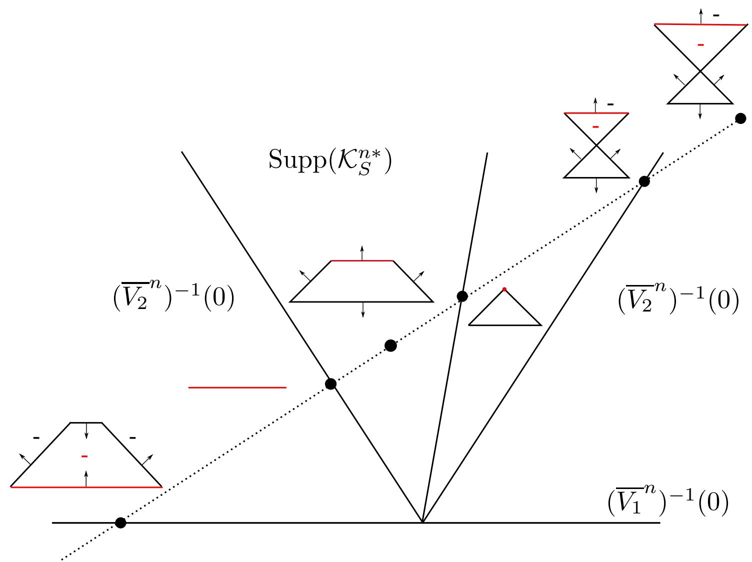

Figure 6: The minus sign outside of the polygons indicates edges with negative algebraic length, while the minus sign inside a polygon indicates a negative $\overline{V_2}^n$. The dotted line is the extension of the segment between the segment and the triangle.

We can endow $\partial\mathscr{O}\mathscr{S}hape^{n*}$ with the visibility metric from $[B^n]$: the distance between $a, b \in \partial\mathscr{O}\mathscr{S}hape^{n*}$, denoted by $<_B(a,b)$, is the angle (with value in $[0,\pi]$) between the two lines c_a and c_b from $[B^n]$ and with endpoints a and b respectively. But clearly the element of $O(n)$ sending the line a to the line b is also a $d_{\mathscr{O}\mathscr{S}^n}$-isometry sending c_a to c_b. In turn, $\partial\mathscr{O}\mathscr{S}hape^{n*}$ endowed with the visibility metric is isometric to $P^{n-1}(\mathbb{R})$ endowed with its round metric. From [9, Proposition II.9.2], $<_B\colon \partial\mathscr{O}\mathscr{S}hape^{n*} \times \partial\mathscr{O}\mathscr{S}hape^{n*} \to \mathbb{R}$ is continuous for the classical topology on $\partial\mathscr{O}\mathscr{S}hape^{n*}$. Hence for this topology, $\partial\mathscr{O}\mathscr{S}hape^{n*}$ is homeomorphic to $P^{n-1}(\mathbb{R})$.

Remark 4.28. The smallest vector space containing $\mathrm{Supp}(\mathcal{K}^n)$ as a convex cone is the vector space spanned by the cone:

$$\mathrm{Sonic}^n = \{h - k | h, k \in \mathrm{Supp}(\mathcal{K}^n)\} ,$$

the space of *n-dimensional hedgehogs*. See [32, 9.6], [33] and the references therein for more informations. Let us say that the name was coined in [22], although they previously appeared in the literature under different names, see [29]. If $h \in \mathrm{Sonic}^n$, there is a way to associate a geometric object in $\mathbb{R}^n$, see [33, 26]. See figures 4, 5 and 6 for illustration.

A description of Sonic^2 in $C^0(\mathbb{S}^1)$ is contained in [26]. But Sonic^n is not complete for any reasonable norm on it —it contains $C^2(\mathbb{S}^{n-1})$, so it is dense in both $H^1(\mathbb{S}^{n-1})$ and $C^0(\mathbb{S}^{n-1})$ endowed with their classical norms.

Partial results were achieved in this setting (mostly in the regular case) in [23, 24, 25].

Remark 4.29 (Plane convex bodies)**.** As a particular example, in the plane $\mathbb{R}^2$ we have

$$d_{\mathscr{O}\mathscr{S}^2}([K],[B^2]) = \mathrm{argch}\left(\frac{\mathrm{per}(K)}{2\sqrt{\pi\,\mathrm{vol}_2(K)}}\right) ,$$

that is coherent with the isoperimetric inequality (we denote by $\mathrm{per}(K)$ is the perimeter of K). In the plane, all the preceding results can be proven directly from Wirtinger's inequality, as the intrinsic area is given by

$$V_2(K) = \frac{1}{2}\int_0^{2\pi} \mathrm{Supp}(K)^2 - (\mathrm{Supp}(K)')^2 \ ,$$

where a support function defined on the unit circle is considered as a 2π-periodic function on $\mathbb{R}$. See the last chapter of [14] for more details.

5 The space of shapes $\mathscr{S}hape^{n*}$

In this section, we will investigate $\mathscr{S}hape^{n*}$, the quotient of $\mathscr{O}\mathscr{S}hape^{n*}$ by the action of linear isometries $O(n)$ of the Euclidean space $\mathbb{R}^n$: $\mathscr{S}hape^{n*}$ is the space of convex bodies in $\mathbb{R}^n$ (not reduced to points or segments) up to Euclidean similarities. We will also study its quotient metric $d_{\mathscr{S}^n}$.

In particular we obtain that, as a quotient of a CBB(-1) space, $(\mathscr{S}hape^{n*}, d_{\mathscr{S}^n})$ is CBB(-1). Although this is a well-known fact (see Section 10.2.2 in [10]), for a matter of presentation we give a complete argument.

5.1 Metric spaces with bounded curvature

Consider the hyperbolic plane $\mathbb{H}^2$, with its hyperbolic metric d_h. Let ABC be a triangle in $\mathbb{H}^2$, and let us define $a = d_h(B,C)$, $b = d_h(A,C)$ and $c = d_h(A,B)$. Let $\mathrm{mid}(a,b,c)$ be the distance between C and the midpoint of the shortest path AB. For futur references, let us note that the function mid is increasing in a and b: using two times the hyperbolic cosine law, one gets

$$\cosh(\mathrm{mid}(a,b,c)) = \frac{\cosh a + \cosh b}{2\cosh(c/2)} \ .$$

Definition 5.1. *We say that a metric space (X,d) is a CAT(-1) (resp. CBB(-1)) space if (X,d) is geodesic and, for any triangle with length sides a,b,c and vertices A,B,C of (X,d), if m is the middle point of the shortest path AB, then*

$$d(C,m) \leq \mathrm{mid}(a,b,c)\ (\text{resp. } d(C,m) \geq \mathrm{mid}(a,b,c)) \ .$$

Note that the condition about the triangles in Definition 5.1 above is global, and that we don't require completeness of (X,d).

5.2 Properties of a quotient metric

Let (X,d) be a metric space, and let O be a subgroup of its isometry group. Assume that O is endowed with a topology for which it is compact, and such that the maps $A \in O \mapsto A(x) \in X$ are continuous for every $x \in X$. Let $\bar{X}$ be the quotient space $\bar{X} = X/O$. For $x \in X$, we denote by $\bar{x}$ the equivalent class. Let $\bar{d} : \bar{X} \times \bar{X} \to \mathbb{R}$ be the following function:

$$\bar{d}(\bar{x},\bar{y}) := \inf_{A,B\in O} d(A(x),B(y)) \ .$$

Since O is a group of isometries we have $d(A(x),B(y)) = d(x,A^{-1}B(y))$, so $\bar{d}(\bar{x},\bar{y}) = \inf_{A\in O} d(x,A(y))$. Moreover, by compactness of O and continuity of the map $A \in O \mapsto A(y) \in X$ we have

$$\bar{d}(\bar{x},\bar{y}) = \min_{A\in O} d(x,A(y)) \ .$$

Lemma 5.2. *The function $\bar{d}$ is a metric on $\bar{X}$.*

Proof. The symmetry property and the fact that $\bar{d}$ separates points are obvious. Now let $\bar{x}, \bar{y}, \bar{z} \in \bar{X}$, and let $A, B \in O$ such that $\bar{d}(\bar{x}, \bar{y}) = d(x, A(y))$ and $\bar{d}(\bar{y}, \bar{z}) = d(y, B(z))$. Then

$$\begin{aligned} \bar{d}(\bar{x}, \bar{z}) &\leq d(x, AB(z)) \leq d(x, A(y)) + d(A(y), AB(z)) \\ &= \bar{d}(\bar{x}, \bar{y}) + d(y, B(z)) = \bar{d}(\bar{x}, \bar{y}) + \bar{d}(\bar{y}, \bar{z}) \ . \end{aligned}$$

□

Let us denote by $p : X \to \bar{X}$ the projection map. It may exist an element x of X such that for a continuous path $A_t \in O$ we have $A_t(x) = x$ (for example this will be the case with $X = \mathscr{O}\mathscr{S}hape^{n*}$ and $x = [B^n]$). In this case, p is not locally injective, so in general p is not a covering map. Nevertheless, the following holds.

Lemma 5.3. *The topology given by $\bar{d}$ on $\bar{X}$ is the quotient topology.*

Proof. We need to show that a set $\bar{\Omega} \subset \bar{X}$ is open (for the topology given by $\bar{d}$) if and only if $p^{-1}(\bar{\Omega}) \subset X$ is an open set.

First, let us observe that the map p is continuous: indeed, if x_n in X converges to x for d, then we have $0 \leq \bar{d}(\bar{x}_n, \bar{x}) \leq d(x_n, x)$, hence $\bar{x}_n$ converge to $\bar{x}$. This shows that if $\bar{\Omega} \subset \bar{X}$ is open, then $p^{-1}(\bar{\Omega}) \subset X$ is an open set.

On the other hand, assume that $\bar{\Omega} \subset \bar{X}$ is a set such that $p^{-1}(\bar{\Omega})$ is an open set of X. We want to show that $\bar{\Omega}$ is open. Let $\bar{x} \in \bar{\Omega}$, and let $x \in X$ be such that $p(x) = \bar{x}$. We define the distance to $X - p^{-1}(\bar{\Omega})$ as the function

$$y \in X \mapsto d(y, X - p^{-1}(\bar{\Omega})) := \inf_{z \in X - p^{-1}(\bar{\Omega})} d(y, z)$$

(note that if $X - p^{-1}(\bar{\Omega}) = \emptyset$, then $\bar{X} = p(X) = p(p^{-1}(\bar{\Omega})) \subset \bar{\Omega}$, so $\bar{\Omega} = \bar{X}$ is an open set).

This function is continuous (indeed it is $1-$Lipschitz), hence the function

$$A \ni O \mapsto d(A(x), X - p^{-1}(\bar{\Omega}))$$

is also continuous. Since O is compact, this function attains its minimum. And this minimum can not be zero: indeed we have $p(A(x)) = p(x) = \bar{x} \in \bar{\Omega}$, so $A(x) \notin X - p^{-1}(\bar{\Omega})$, which gives $d(A(x), X - p^{-1}(\bar{\Omega})) > 0$ (note that $X - p^{-1}(\bar{\Omega})$ is a closed set). So there exists $r > 0$ such that

$$d(A(x), y) \geq r \text{ for every } A \in O \text{ and every } y \in X - p^{-1}(\bar{\Omega}) \ . \tag{5.1}$$

This shows that $B_{\bar{X}}(\bar{x}, r)$ (which is the ball in $(\bar{X}, \bar{d})$ with center $\bar{x}$ and radius r) is included in $\bar{\Omega}$. Indeed, let $\bar{y} \in B_{\bar{X}}(\bar{x}, r)$, and let $y \in X$ be such that $p(y) = \bar{y}$: there exists $A \in O$ such that $\bar{d}(\bar{x}, \bar{y}) = d(A(x), y) < r$, and equation (5.1) shows that $y \in p^{-1}(\bar{\Omega})$, that is $\bar{y} = p(y) \in \bar{\Omega}$.

□

The aim of the rest of this section is to prove the following.

Proposition 5.4. *The following properties hold:*

1. *If d is proper, then $\bar{d}$ is proper.*
2. *If d is a geodesic metric, then $\bar{d}$ is a geodesic metric.*
3. *If (X, d) is CBB(-1), then $(\bar{X}, \bar{d})$ is CBB(-1).*

Proof of property 1 in Proposition 5.4. Suppose that (X, d) is a proper metric space, and let $(\bar{x}_i)_{i \in \mathbb{N}}$ be a bounded sequence in $(\bar{X}, \bar{d})$. There are $A_i \in O$ such that $(A_i(x_i))_{i \in \mathbb{N}}$ is a bounded sequence in (X, d). Since (X, d) is proper, up to extract a subsequence, there exists $y \in X$ such that $d(A_i(x_i), y) \to 0$. As $\bar{d}(\bar{x}_i, \bar{y}) \leq d(A_i(x_i), y)$, we have $\bar{d}(\bar{x}_i, \bar{y}) \to 0$. □

Property 2 in Proposition 5.4 is a direct consequence of the

Lemma 5.5. *Let $x, y \in X$, and let $A \in O$ be such that $\bar{d}(\bar{x}, \bar{y}) = d(x, A(y))$. Suppose that γ is a shortest path between x and $A(y)$. Then the projection $\bar{\gamma} = p \circ \gamma$ is a shortest path between $\bar{x}$ and $\bar{y}$. Moreover the projection is an isometry from γ to $\bar{\gamma}$.*

Proof. Let us suppose that $\gamma : [0, 1] \to X$ is affinely parametrized. Then for any $0 \leq s \leq t \leq 1$,

$$\bar{d}(\bar{\gamma}(s), \bar{\gamma}(t)) \leq d(\gamma(s), \gamma(t)) = (t - s)d(x, A(y)) = (t - s)\bar{d}(\bar{x}, \bar{y}) \ .$$

Using three times this inequality we obtain

$$\begin{aligned} \bar{d}(\bar{x}, \bar{y}) &\leq \bar{d}(\bar{\gamma}(0), \bar{\gamma}(s)) + \bar{d}(\bar{\gamma}(s), \bar{\gamma}(t)) + \bar{d}(\bar{\gamma}(t), \bar{\gamma}(1)) \\ &\leq (s + (t - s) + (1 - t))\bar{d}(\bar{x}, \bar{y}) = \bar{d}(\bar{x}, \bar{y}) \ . \end{aligned}$$

All these inequalities are equalities, so in particular $\bar{d}(\bar{\gamma}(s), \bar{\gamma}(t)) = (t - s)\bar{d}(\bar{x}, \bar{y})$. □

In general, this is not true that every shortest path in the quotient space is obtained as the projection of a shortest path. To prove property 3 in Proposition 5.4 we need the following Lemma:

Lemma 5.6. *Assume that d is geodesic (hence $\bar{d}$ is also geodesic). If $\bar{w} \in \bar{X}$ is the midpoint of a shortest path between $\bar{x} \in \bar{X}$ and $\bar{y} \in \bar{X}$, then there exists $A, B \in O$ such that:*

- *$\bar{d}(\bar{x}, \bar{y}) = d(x, A(y))$ and*
- *$B(w)$ is the midpoint of a shortest path joining x and $A(y)$ in X.*

Proof. Let $B \in O$ such that $\bar{d}(\bar{x}, \bar{w}) = d(x, B(w))$, and $A \in O$ such that $\bar{d}(\bar{y}, \bar{w}) = d(y, A^{-1}B(w))$. We have

$$\begin{aligned} \bar{d}(\bar{x}, \bar{y}) &\leq d(x, A(y)) \leq d(x, B(w)) + d(B(w), A(y)) \\ &= \bar{d}(\bar{x}, \bar{w}) + d(A^{-1}B(w), y) = \bar{d}(\bar{x}, \bar{w}) + \bar{d}(\bar{y}, \bar{w}) = \bar{d}(\bar{x}, \bar{y}) \ . \end{aligned}$$

Hence all these inequalities are equalities and this ends the proof. □

Proof of property 3 in Proposition 5.4. Suppose that (X, d) is a CBB(-1) space. Then in particular (X, d) and $(\bar{X}, \bar{d})$ are geodesic metric spaces.

Let T be a geodesic triangle in $\bar{X}$ with vertices $\bar{x}\bar{y}\bar{z}$, and let $\bar{w}$ be the midpoint of the geodesic between $\bar{x}$ and $\bar{y}$. In order to show that $(\bar{X}, \bar{d})$ is CBB(-1) we need to prove the following inequality:

$$\bar{d}(\bar{z}, \bar{w}) \geq \mathrm{mid}(\bar{d}(\bar{y}, \bar{z}), \bar{d}(\bar{x}, \bar{z}), \bar{d}(\bar{x}, \bar{y})) \ . \tag{5.2}$$

By Lemma 5.6, there exists $A, B \in O$ such that $\bar{d}(\bar{x}, \bar{y}) = d(x, A(y))$, and $B(w)$ is the midpoint of a shortest path L joining x and $A(y)$ in X. Let $C \in O$ be such that $\bar{d}(\bar{z}, \bar{w}) = d(C(z), B(w))$.

Now consider the geodesic triangle $xA(y)C(z)$ in X (the shortest path between x and $A(y)$ is L). Since (X, d) is CBB(-1), and $B(w)$ is the midpoint of the geodesic between x and $A(y)$, we have

$$\bar{d}(\bar{z}, \bar{w}) = d(C(z), B(w)) \geq \mathrm{mid}(a, b, c) \ , \tag{5.3}$$

where

$$\begin{aligned} a &= d(A(y), C(z)) \geq \bar{d}(\bar{y}, \bar{z}) \ , \\ b &= d(x, C(z)) \geq \bar{d}(\bar{x}, \bar{z}) \ , \\ c &= d(x, A(y)) = \bar{d}(\bar{x}, \bar{y}) \ . \end{aligned} \tag{5.4}$$

This proves (5.2) since the function mid is increasing in a and b. □

5.3 The space of shapes

Let $\mathscr{S}hape^{n*}$ be the quotient of $\mathscr{O}\mathscr{S}hape^{n*}$ by linear isometries of the Euclidean space $\mathbb{R}^n$: the action of $O(n)$ on $\mathscr{O}\mathscr{S}hape^{n*}$ is defined by $\Phi[K] := [\Phi K]$. For $K \in \mathcal{K}^{n*}$, we will denote by $[\![K]\!]$ the set of convex bodies differing from K by positive homotheties and Euclidean isometries.

Since V_2 is $O(n)-$invariant, we have $d_{\mathscr{O}\mathscr{S}^n}(\Phi[K_1], \Phi[K_2]) = d_{\mathscr{O}\mathscr{S}^n}([K_1],[K_2])$, so $O(n)$ acts by isometries on $\mathscr{O}\mathscr{S}hape^{n*}$.

Fact 5.7. *Let $K \in \mathscr{O}\mathscr{S}\mathrm{hape}^{n*}$. The map $O(n) \ni \Phi \mapsto \Phi[K] \in \mathscr{O}\mathscr{S}\mathrm{hape}^{n*}$ is continuous.*

Proof. We have $\mathrm{Supp}(\Phi K)(x) = \mathrm{Supp}(K)(\Phi^{-1}(x))$. In particular, if $\Phi_n \to \Phi$ in $O(n)$, then for any $x \in \mathbb{S}^{n-1}$, $\mathrm{Supp}(\Phi_n K)(x)$ converges to $\mathrm{Supp}(\Phi K)(x)$. The result follows from Lemma 4.5 and Corollary 4.20. □

This fact shows that we can apply the results of the previous section, for the action of the (compact) group $O(n)$ on $(\mathscr{O}\mathscr{S}hape^{n*}, d_{\mathscr{O}\mathscr{S}^n})$. Let us introduce

$$d_{\mathscr{S}^n}([\![K_1]\!],[\![K_2]\!]) = \inf_{\Phi,\Phi' \in O(n)} d_{\mathscr{O}\mathscr{S}^n}(\Phi[K_1], \Phi'[K_2]) \ . \tag{5.5}$$

We have $d_{\mathscr{S}^n}([\![K_1]\!],[\![K_2]\!]) = \min_{\Phi \in O(n)} d_{\mathscr{O}\mathscr{S}^n}([K_1], \Phi[K_2])$. The previous section gives the following.

Proposition 5.8. *$(\mathscr{S}\mathrm{hape}^{n*}, d_{\mathscr{S}^n})$ is a CBB(-1) proper geodesic metric space.*

5.4 Non-uniqueness of shortest paths in $\mathscr{S}hape^{n*}$

The aim of this section is to prove that shortest paths are not unique in $\mathscr{S}hape^{n*}$. Obviously, since $\mathscr{S}hape^{2*}$ isometrically embedds in $\mathscr{S}hape^{n*}$ for $n \geq 2$, this is sufficient to prove this property for $n = 2$. Hence in this section we consider convex bodies in $\mathbb{R}^2$.

Let K be the intersection of the half-space $[0,\infty) \times \mathbb{R}$ with the ellipse with center 0, width $2\sqrt{2}$ and height $\frac{2}{\sqrt{2}}$. The support function of K is a function on $\mathbb{S}^1$, and with the parametrization $x = (\cos s, \sin s) \in \mathbb{S}^1$, for $s \in [0, 2\pi]$, we will actually define the support function k of K on $[0, 2\pi]$. Namely,

$$k(s) = \sqrt{2\cos^2 s + \frac{1}{2}\sin^2 s} \text{ for } s \in [-\frac{\pi}{2}, \frac{\pi}{2}], \text{ and } k(s) = \frac{1}{\sqrt{2}}|\sin s| \text{ for } s \in [\frac{\pi}{2}, \frac{3\pi}{2}] \ .$$

Let $(\beta, 0)$ be the Steiner point of K, and let $\alpha = V_1(K) = \frac{1}{2}\int_0^{2\pi} k \simeq 2.4$. Then the convex body $K_1 = \alpha^{-1}K + (-\alpha^{-1}\beta, 0)$ has Steiner point 0, and $V_1(K_1) = 1$: hence $K_1 \in \mathcal{K}^{2*}_{SV_1}$. Its support function $k_1 \in \mathrm{Supp}(\mathcal{K}^{2*}_{SV_1})$ is given by

$$k_1(s) = \alpha^{-1}\left(\sqrt{2\cos^2 s + \frac{1}{2}\sin^2 s} - \beta\cos s\right) \text{ for } s \in [\frac{-\pi}{2}, \frac{\pi}{2}]$$

and

$$k_1(s) = \alpha^{-1}\left(\frac{1}{\sqrt{2}}|\sin s| - \beta\cos s\right) \text{ for } s \in [\frac{\pi}{2}, \frac{3\pi}{2}] \ .$$

Let K_2 be the rectangle $[-\frac{2}{5}, \frac{2}{5}] \times [-\frac{1}{10}, \frac{1}{10}]$. Obviously, 0 is the Steiner point of K_2. Its support function is defined for any $s \in [0, 2\pi]$ by

$$k_2(s) = \frac{2}{5}|\cos s| + \frac{1}{10}|\sin s| \ ,$$

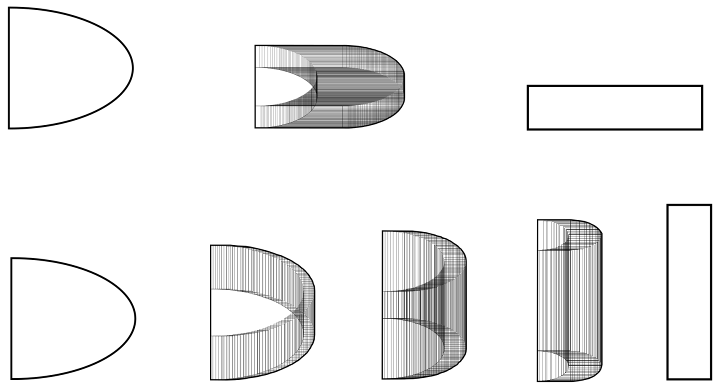

Figure 7: The convex body $\frac{1}{2}\alpha^{-1}K + \frac{1}{2}K_2$ (middle of the upper line) is not the image by a rotation and a translation of $(1-t)\alpha^{-1}K + tR_{\frac{\pi}{2}}(K_2)$ (represented on the bottom line for $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$).

and since $K_2 = [-\frac{2}{5}, \frac{2}{5}] \times \{0\} + \{0\} \times [-\frac{1}{10}, \frac{1}{10}]$, we have $V_1(K_2) = \text{length}([-\frac{2}{5}, \frac{2}{5}]) + \text{length}([-\frac{1}{10}, \frac{1}{10}]) = 1$. Hence $K_2 \in \mathcal{K}^{2*}_{SV_1}$ and $k_2 \in \text{Supp}(\mathcal{K}^{2*}_{SV_1})$.

Let $[\![K_1]\!]$ and $[\![K_2]\!]$ be the corresponding equivalent classes in $\mathscr{S}hape^{2*}$. Since K_2 is invariant by the symmetry with respect to the horizontal line, the distance between $[\![K_1]\!]$ and $[\![K_2]\!]$ is given by

$$d_{\mathscr{S}^2}([\![K_1]\!], [\![K_2]\!]) = \min_{\theta \in \mathbb{R}} d_{\mathscr{O}\mathscr{S}^2}([K_1], R_\theta[K_2]) \ ,$$

where we denote by R_θ the rotation of angle θ in $\mathbb{R}^2$. We will prove the following:

Proposition 5.9. *The minimum is obtained for $\theta = 0$ and $\theta = \frac{\pi}{2}$, that is we have*

$$d_{\mathscr{S}^2}([\![K_1]\!], [\![K_2]\!]) = d_{\mathscr{O}\mathscr{S}^2}([K_1], [K_2]) = d_{\mathscr{O}\mathscr{S}^2}([K_1], R_{\frac{\pi}{2}}[K_2]) \ .$$

This Proposition is sufficient to prove the non-uniqueness of shortest paths in $\mathscr{S}hape^{2*}$. Indeed, Lemma 5.5 shows that the projections of the shortest paths in $\mathscr{O}\mathscr{S}hape^{2*}$ between $[K_1]$ and $[K_2]$, and between $[K_1]$ and $R_{\frac{\pi}{2}}[K_2]$, are again shortest paths in $\mathscr{S}hape^{2*}$. But these two shortest paths are different: the first shortest path contains the point $[\![\frac{1}{2}K_1 + \frac{1}{2}K_2]\!]$, and this point is not on the second shortest path $t \mapsto [\![(1-t)K_1 + tR_{\frac{\pi}{2}}(K_2)]\!]$: $\frac{1}{2}K_1 + \frac{1}{2}K_2$ is not the image by a rotation of $(1-t)K_1 + tR_{\frac{\pi}{2}}(K_2)$, which is equivalent to say that $\frac{1}{2}\alpha^{-1}K + \frac{1}{2}K_2$ is not the image by a rotation and a translation of $(1-t)\alpha^{-1}K + tR_{\frac{\pi}{2}}(K_2)$. See Figure 7.

Since $R_\pi[K_2] = [K_2]$, to compute the minimum this is sufficient to consider $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, let T be the symmetry with respect to the x axis: we have $T[K_1] = [K_1]$, hence we have

$$d_{\mathscr{O}\mathscr{S}^2}([K_1], R_\theta[K_2]) = d_{\mathscr{O}\mathscr{S}^2}(T[K_1], R_\theta[K_2]) = d_{\mathscr{O}\mathscr{S}^2}([K_1], T \circ R_\theta[K_2]) = d_{\mathscr{O}\mathscr{S}^2}([K_1], R_{-\theta}[K_2]) \ .$$

This shows that in fact we need only to consider $\theta \in [0, \frac{\pi}{2}]$.

Let k_2^θ be the support function of $R_\theta[K_2]$, that is $k_2^\theta(s) = k_2(s-\theta)$. We have

$$\cosh(d_{\mathscr{C}\mathscr{S}^2}([K_1], R_\theta[K_2])) = \frac{V_2(k_1, k_2^\theta)}{\sqrt{V_2(k_1)V_2(k_2^\theta)}} = \frac{f(\theta)}{2\sqrt{V_2(k_1)V_2(k_2)}} \ ,$$

where we denote by $f(\theta)$ the function defined by

$$f(\theta) = \int_0^{2\pi} (k_1(s)k_2(s-\theta) - k_1'(s)k_2'(s-\theta))\mathrm{d}s \ .$$

Proposition 5.9 is a direct consequence of the following Lemma:

Lemma 5.10. *On $[0, \frac{\pi}{2}]$, f attains its minimum at the points $\theta = 0$ and $\theta = \frac{\pi}{2}$.*

Proof. Fix $\theta \in (0, \frac{\pi}{2})$, and consider the function $s \mapsto k_1(s)k_2'(s-\theta)$. This function is piecewise $\mathcal{C}^1$, but is not continuous: the function $k_2'(s-\theta)$ has jumps, with height $\frac{1}{5}$ at the points $s=\theta$ and $s = \pi+\theta$, and with height $\frac{4}{5}$ at the points $s = \frac{\pi}{2}+\theta$ and $s = \frac{3\pi}{2}+\theta$. Hence we have

$$\int_0^{2\pi} (k_1(s)k_2'(s-\theta))'\mathrm{d}s = -\frac{1}{5}k_1(\theta) - \frac{1}{5}k_1(\pi+\theta) - \frac{4}{5}k_1(\frac{\pi}{2}+\theta) - \frac{4}{5}k_1(\frac{3\pi}{2}+\theta)$$
$$= -\frac{1}{5\alpha}\sqrt{2\cos^2\theta + \frac{1}{2}\sin^2\theta} - \frac{4}{5\alpha}\sqrt{2\sin^2\theta + \frac{1}{2}\cos^2\theta} - \frac{1}{5\sqrt{2}\alpha}\sin\theta - \frac{4}{5\sqrt{2}\alpha}\cos\theta \ .$$

The equality $(k_1k_2')' = k_1'k_2' + k_1k_2''$ gives $-k_1'k_2' = k_1k_2'' - (k_1k_2')'$, so

$$-\int_0^{2\pi} k_1'(s)k_2'(s-\theta)\mathrm{d}s = \int_0^{2\pi} (k_1(s)k_2''(s-\theta) - (k_1(s)k_2'(s-\theta))')\mathrm{d}s \ ,$$

and since $k_2(s-\theta) + k_2''(s-\theta) = 0$ for almost every $s \in [0, 2\pi]$ we finally obtain

$$\begin{aligned}
f(\theta) &= \int_0^{2\pi} (k_1(s)k_2(s-\theta) - k_1'(s)k_2'(s-\theta))\mathrm{d}s \\
&= \int_0^{2\pi} (k_1(s)(k_2(s-\theta) + k_2''(s-\theta)) - (k_1(s)k_2'(s-\theta))')\mathrm{d}s \\
&= \frac{1}{5\alpha}\sqrt{2\cos^2\theta + \frac{1}{2}\sin^2\theta} + \frac{4}{5\alpha}\sqrt{2\sin^2\theta + \frac{1}{2}\cos^2\theta} + \frac{1}{5\sqrt{2}\alpha}\sin\theta + \frac{4}{5\sqrt{2}\alpha}\cos\theta \ .
\end{aligned}$$

We easily check that $f(0) = f(\frac{\pi}{2}) = \frac{\sqrt{2}}{\alpha}$ (the parameters of the ellipse and the segment have been chosen so that this property holds). And a direct computation shows that $f'(0) = \frac{1}{5\sqrt{2}\alpha} > 0$ and $f'(\frac{\pi}{2}) = -\frac{4}{5\sqrt{2}\alpha} < 0$. Moreover, let $g : [0,1] \to [0, \infty)$ be defined by

$$g(u) = \frac{1}{5\alpha}\sqrt{\frac{3}{2}u + \frac{1}{2}} + \frac{4}{5\alpha}\sqrt{2 - \frac{3}{2}u} + \frac{1}{5\sqrt{2}\alpha}\sqrt{1-u} + \frac{4}{5\sqrt{2}\alpha}\sqrt{u} \ :$$

with the identity $\cos^2 + \sin^2 = 1$ we easily check that $g(\cos^2\theta) = f(\theta)$ for any $\theta \in [0, \frac{\pi}{2}]$. Hence $f'(\theta) = -2g'(\cos^2\theta)\sin\theta\cos\theta$. But g is strictly concave, hence g' has at most one zero on $[0,1]$, hence f' has also at most one zero on $(0, \frac{\pi}{2})$. And this ends the proof: if the minimum of f on $[0, \frac{\pi}{2}]$ was attained at a point $\theta \notin \{0, \frac{\pi}{2}\}$, since $f'(0) > 0$ and $f'(\frac{\pi}{2}) < 0$, f' would have at least 3 zeros on $(0, \frac{\pi}{2})$, and that is impossible. ☐

5.5 Embedding of hyperbolic planes

Trivially, for any $\Phi \in O(n)$ we have $\Phi[B^n] = [B^n]$. Apart from the fact that the action of $O(n)$ on $\mathscr{OS}hape^{n*}$ is not proper, this says that for any $[K] \in \mathscr{OS}hape^{n*}$,

$$d_{\mathscr{S}^n}([\![K]\!], [\![B^n]\!]) = d_{\mathscr{OS}^n}([K], [B^n]) \ . \tag{5.6}$$

From this we first deduce the following fact.

Fact 5.11 (Uniqueness of shortest paths starting from B^n)**.** *Let $[\![K]\!] \in \mathscr{S}\mathrm{hape}^{n*}$. Then there is a* unique *shortest path from $[\![B^n]\!]$ to $[\![K]\!]$, which is the projection of the shortest path in $\mathscr{OS}\mathrm{hape}^{n*}$ between $[B^n]$ and $[K]$.*

Proof. Let $\bar{\delta} : [0, d_{\mathscr{S}^n}([\![B^n]\!], [\![K]\!])] \to \mathscr{S}hape^{n*}$ be an arc-length parametrized shortest path between $[\![B^n]\!]$ and $[\![K]\!]$, and let $[\delta(t)] \in \mathscr{OS}hape^{n*}$ be such that $\bar{\delta}(t) = [\![\delta(t)]\!]$. Let $t \mapsto [\gamma(t)]$ be the (unique) arc-length parametrized shortest path in $\mathscr{OS}hape^{n*}$ between $[B^n]$ and $[K]$: we want to show that $[\![\delta(t)]\!] = [\![\gamma(t)]\!]$.

For any $t \in [0, d_{\mathscr{S}^n}([\![B^n]\!], [\![K]\!])]$, let $\Phi_t \in O(n)$ be such that

$$d_{\mathscr{S}^n}([\![K]\!], [\![\delta(t)]\!]) = d_{\mathscr{OS}^n}([K], \Phi_t[\delta(t)]) \ .$$

Since $t \mapsto [\![\delta(t)]\!]$ is a geodesic in $\mathscr{S}hape^{n*}$ we have

$$\begin{aligned} d_{\mathscr{OS}^n}([B^n], \Phi_t[\delta(t)]) + d_{\mathscr{OS}^n}(\Phi_t[\delta(t)], [K]) &= d_{\mathscr{S}^n}([\![B^n]\!], [\![\delta(t)]\!]) + d_{\mathscr{S}^n}([\![\delta(t)]\!], [\![K]\!]) \\ &= d_{\mathscr{S}^n}([\![B^n]\!], [\![K]\!]) = d_{\mathscr{OS}^n}([B^n], [K]) \ . \end{aligned}$$

Hence $\Phi_t[\delta(t)]$ is on the shortest path between $[B^n]$ and $[K]$ in $\mathscr{OS}hape^{n*}$. Moreover we have $d_{\mathscr{OS}^n}([B^n], \Phi_t[\delta(t)]) = d_{\mathscr{S}^n}([\![B^n]\!], [\![\delta(t)]\!]) = t$ (the geodesic $t \mapsto [\![\delta(t)]\!]$ is arc-length parametrized), so $\Phi_t[\delta(t)] = [\gamma(t)]$ (remember that the geodesic $t \mapsto [\gamma(t)]$ is also arc-length parametrized). Finally this gives $[\![\delta(t)]\!] = [\![\gamma(t)]\!]$. □

In turn, we can construct totally geodesic hyperbolic surfaces in $\mathscr{S}hape^{n*}$.

Proposition 5.12. *Let $[\![P]\!], [\![Q]\!] \in \mathscr{S}\mathrm{hape}^{n*}$ be such that $[\![P]\!], [\![Q]\!]$ and $[\![B^n]\!]$ are three different points. Let $A \in O(n)$ be such that $d_{\mathscr{S}^n}([\![P]\!], [\![Q]\!]) = d_{\mathscr{OS}^n}([P], A[Q])$. Then the projection $\mathscr{OS}\mathrm{hape}^{n*} \to \mathscr{S}\mathrm{hape}^{n*}$, when restricted to the (plain) geodesic triangle with vertices $[B^n], [P]$ and $A[Q]$, is an isometry onto its image.*

Proof. Without loss of generality, we may assume that A is the identity (that is, $d_{\mathscr{S}^n}([\![P]\!], [\![Q]\!]) = d_{\mathscr{OS}^n}([P], [Q])$). Let $[K_1]$ and $[K_2]$ be in the geodesic triangle with vertices $[B^n], [P]$ and $[Q]$: since geodesics in $\mathscr{OS}hape^{n*}$ are convex combinations, we can write

$$[K_1] = [\alpha_1 B^n + \beta_1 P + \gamma_1 Q] \text{ and } [K_2] = [\alpha_2 B^n + \beta_2 P + \gamma_2 Q] \ ,$$

where the $\alpha_i, \beta_i, \gamma_i$ are non-negative real numbers, with $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1$. We want to prove that $d_{\mathscr{S}^n}([\![K_1]\!], [\![K_2]\!]) = d_{\mathscr{OS}^n}([K_1], [K_2])$, which means that for any $\Phi \in O(n)$ we have $d_{\mathscr{OS}^n}([K_1], [K_2]) \leq d_{\mathscr{OS}^n}([K_1], \Phi[K_2])$. Since V_2 is $O(n)-$invariant, we only need to show that

$$V_2(K_1, K_2) \leq V_2(K_1, \Phi(K_2)) \tag{5.7}$$

(K_1 and K_2 denote two convex bodies in the equivalent classes $[K_1]$ and $[K_2]$). We have

$$V_2(K_1, K_2) = \alpha_1\alpha_2 V_2(B^n) + \alpha_1\beta_2 V_2(B^n, P) + \alpha_1\gamma_2 V_2(B^n, Q)$$

$$+\beta_1\alpha_2 V_2(P, B^n) + \beta_1\beta_2 V_2(P) + \beta_1\gamma_2 V_2(P, Q)$$

$$+\gamma_1\alpha_2 V_2(Q, B^n) + \gamma_1\beta_2 V_2(Q, P) + \gamma_1\gamma_2 V_2(Q) \ .$$

Moreover $\Phi(K_2) = \alpha_2 B^n + \beta_2\Phi(P) + \gamma_2\Phi(Q)$, hence

$$V_2(K_1, \Phi(K_2)) = \alpha_1\alpha_2 V_2(B^n) + \alpha_1\beta_2 V_2(B^n, \Phi(P)) + \alpha_1\gamma_2 V_2(B^n, \Phi(Q))$$
$$+\beta_1\alpha_2 V_2(P, B^n) + \beta_1\beta_2 V_2(P, \Phi(P)) + \beta_1\gamma_2 V_2(P, \Phi(Q))$$
$$+\gamma_1\alpha_2 V_2(Q, B^n) + \gamma_1\beta_2 V_2(Q, \Phi(P)) + \gamma_1\gamma_2 V_2(Q, \Phi(Q))\ .$$

And we obviously have $V_2(B^n, P) = V_2(B^n, \Phi(P))$ and $V_2(B^n, Q) = V_2(B^n, \Phi(Q))$. Moreover, the Alexandrov–Fenchel inequality (1.6) gives $V_2(P) = \sqrt{V_2(P)V_2(\Phi(P))} \leq V_2(P, \Phi(P))$, and $V_2(Q) = \sqrt{V_2(Q)V_2(\Phi(Q))} \leq V_2(Q, \Phi(Q))$. And $d_{\mathscr{S}^n}([\![P]\!], [\![Q]\!]) = d_{\mathscr{O}\mathscr{S}^n}([P], [Q])$ gives $V_2(P, Q) \leq V_2(P, \Phi(Q))$ and $V_2(Q, P) \leq V_2(Q, \Phi(P))$. Since all the real numbers $\alpha_i, \beta_i, \gamma_i$ are non-negative, this gives inequality (5.7).

□

5.6 Proof of Theorem 3

Proposition 5.8 and sections 5.4 and 5.5 give part of Theorem 3. It remains to prove the assertion about the boundary of $\mathscr{S}hape^{n*}$. It obviously contains only one point: indeed, the boundary of $\mathscr{O}\mathscr{S}hape^{n*}$ is the set of segments up to translations and homotheties, so the boundary of $\mathscr{S}hape^{n*}$ is the set of segments, up to translations, homotheties and isometries of $\mathbb{R}^n$, and there is only one equivalence class.

5.7 Metrics of non-negative curvature on the sphere $\mathbb{S}^2$

Let $\mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$ be the set of metrics of non-negative curvature on the sphere in the sense of Alexandrov, up to isometries, and with unit area. Let K be a convex body in $\mathbb{R}^3$. Then the induced inner distance on the boundary of K (i.e. the infimum of the length of curves on the boundary between the two points) is isometric to a metric of non-negative curvature on $\mathbb{S}^2$ [2], [10, Theorem 10.2.6]. This gives a well defined map

$$\mathcal{I} : \mathcal{K}^{3*}_{SV_2} \to \mathcal{M}_{\geq 0}(\mathbb{S}^2)^1\ ,$$

where we denote by $\mathcal{K}^{3*}_{SV_2}$ the space of convex bodies in $\mathbb{R}^3$, not reduced to points or segments, with Steiner point at the origin and intrinsic area V_2 equals to one. Remember that the total surface area is two times the intrinsic area, so that for $K \in \mathcal{K}^{3*}_{SV_2}$, $\mathcal{I}(K)$ is a homothety of factor $2^{-1/2}$ of the induced inner distance on the boundary of K. We have the following classical result, see [1], as well as [8] for an alternative proof.

Theorem 5.13 (Alexandrov)**.** *For any metric m of non-negative curvature on the sphere, there exists a convex body K in $\mathbb{R}^3$ such that the induced inner distance on the boundary of K is isometric to m.*

Of course, the induced inner distance does not change if an isometry of the ambient Euclidean space is performed on K. In particular, a translation can be performed on K, such that its Steiner point becomes the origin. Hence Theorem 5.13 implies the following.

Corollary 5.14. *The map $\mathcal{I}$ is surjective.*

Later Pogorelov proved that convex bodies in $\mathbb{R}^3$ are uniquely determined, up to global isometries, by the induced inner distance on their boundary. See [31], as well as [38] for a stronger result.

Theorem 5.15 (Pogorelov)**.** *Let K_1 and K_2 in $\mathcal{K}^{3*}_{SV_2}$ such that $\mathcal{I}(K_1) = \mathcal{I}(K_2)$. Then K_1 and K_2 differ by an element of $O(3)$.*

Note that $\mathrm{Supp}(\mathcal{K}^{3*}_{SV_2}) \subset \mathcal{H}^{\infty}_3$. By Corollary 3.7 and Corollary 4.19, one obtains the following.

Lemma 5.16. *On* $\mathrm{Supp}(\mathcal{K}^{3*}_{SV_2})$*,* $d_{\mathcal{H}}$ *and* d_∞ *induce the same topology.*

Let d_{SV_2} be the distance on $\mathcal{K}^{3*}_{SV_2}$ defined as the pullback of $d_{\mathcal{H}}$ by the map $\mathrm{Supp} : \mathcal{K}^{3*}_{SV_2} \to \mathrm{Supp}(\mathcal{K}^{3*}_{SV_2})$. By construction of the metrics, this is also the pullback of the distance $d_{\mathscr{O}\mathscr{S}^3}$ by the projection map $\mathcal{K}^{3*}_{SV_2} \to \mathscr{O}\mathscr{S}hape^{3*}$. By Remark 4.7, the lemma above can be rephrased as follows.

Lemma 5.17. *On* $\mathcal{K}^{3*}_{SV_2}$*,* d_{SV_2} *and the Hausdorff distance induce the same topology.*

Also, the space $\mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$ is endowed with a natural topology, which is the one of uniform convergence of metric spaces [10, 7.1.5] (which is actually, in this very particular case, the Gromov–Hausdorff topology, [10, Exercise 7.5.14]).

Lemma 5.18. *The map* $\mathcal{I}$ *is continuous for the topology given by* d_{SV_2} *on* $\mathcal{K}^{3*}_{SV_2}$*, and the topology of the uniform convergence on* $\mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$*.*

Proof. By [10, Lemma 10.2.7], $\mathcal{I}$ is continuous when $\mathcal{K}^{3*}_{SV_2}$ is endowed with the Hausdorff topology. Hence Lemma 5.17 proves the claim. □

Let $p : \mathcal{K}^{3*}_{SV_2} \to \mathscr{S}hape^{3*}$ be the map which associates to any element of $\mathcal{K}^{3*}_{SV_2}$ its class in $\mathscr{S}hape^{3*}$. We have $p = p_2 \circ p_1$, where p_1 and p_2 are the following projections:

$$(\mathcal{K}^{3*}_{SV_2}, d_{SV_2}) \xrightarrow{p_1} (\mathscr{O}\mathscr{S}hape^{3*}, d_{\mathscr{O}\mathscr{S}^3}) \xrightarrow{p_2} (\mathscr{S}hape^{3*}, d_{\mathscr{S}^3})\ .$$

The projection p is continuous: indeed, p_1 is continuous because d_{SV_2} is the pullback of $d_{\mathscr{O}\mathscr{S}^3}$ by the map p_1, and p_2 is continuous since the topology on $\mathscr{S}hape^{3*}$ is the quotient topology (Lemma 5.3).

Let us define the map $\bar{\mathcal{I}}$ by the following commutative diagram

$$\begin{array}{ccc} \mathcal{K}^{3*}_{SV_2} & \xrightarrow{\mathcal{I}} & \mathcal{M}_{\geq 0}(\mathbb{S}^2)^1 \\ \downarrow p & \nearrow \bar{\mathcal{I}} & \\ \mathscr{S}hape^{3*} & & \end{array}$$

By Corollary 5.14 and Theorem 5.15, $\bar{\mathcal{I}}$ is well-defined and bijective.

Lemma 5.19. *The map* $\bar{\mathcal{I}}$ *is a homeomorphism.*

Proof. As $\mathcal{I}$ is continuous and as the topology on $\mathscr{S}hape^{3*}$ is the quotient topology (Lemma 5.3), $\bar{\mathcal{I}}$ is continuous. Let us check that $\bar{\mathcal{I}}^{-1}$ is continuous. This is sufficient to show that for every sequence (m_j) of $\mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$ converging to $m \in \mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$, there exists a subsequence (m_{j_k}) such that $(\bar{\mathcal{I}}^{-1}(m_{j_k}))$ converges to $\bar{\mathcal{I}}^{-1}(m)$.

Let (m_j) be a sequence of $\mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$ converging to $m \in \mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$ for the uniform topology, and let $[\![K_j]\!] = \bar{\mathcal{I}}^{-1}(m_j)$. Let us choose representatives K_j of $[\![K_j]\!]$ in $\mathcal{K}^{3*}_{SV_2}$. In particular, $\mathcal{I}(K_j) = m_j$, and the convergence of the m_j implies that the diameters of the ∂K_j are uniformly bounded [10, 7.3.14]. In turn, the Euclidean distances between points on $\partial K_j \subset \mathbb{R}^3$ are uniformly bounded. As the Steiner points of the K_j are at the origin, the K_j are contained in a ball. By the Blaschke selection theorem (Theorem 4.15), there is a converging subsequence K_{j_k} for the Hausdorff topology, hence for d_{SV_2} by Lemma 5.17; we set $K = \lim K_{j_k} \in \mathcal{K}^{3*}_{SV_2}$. Since p is continuous, $[\![K_{j_k}]\!]$ converges to $[\![K]\!]$. As $\bar{\mathcal{I}}$ is continuous, $m_{j_k} = \bar{\mathcal{I}}([\![K_{j_k}]\!])$ converges to $\bar{\mathcal{I}}([\![K]\!])$, so $m = \bar{\mathcal{I}}([\![K]\!])$. Finally this gives that $\bar{\mathcal{I}}^{-1}(m_{j_k}) = [\![K_{j_k}]\!]$ converges to $\bar{\mathcal{I}}^{-1}(m) = [\![K]\!]$. □

More informations about the topology of $\mathcal{M}_{\geq 0}(\mathbb{S}^2)^1$ may be found in [4]. For $n \geq 3$, the induced inner distance on the boundary of a convex body in $\mathbb{R}^n$ is (isometric to) a metric of non-negative curvature on $\mathbb{S}^{n-1}$ in the sense of Alexandrov. But not every such metric on $\mathbb{S}^{n-1}$ arises in this way ([21], [1, 1.9]).

6 The space of all the (oriented) shapes

This section is an opening to the study of spaces of convex bodies, considered without making distinction between dimensions.

For $p \geq 0$ let us denote by $\iota_{n,p}$ the canonical isometric embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+p}$ which is given by $\mathbb{R}^n \simeq \mathbb{R}^n \times \{0\}^p \subset \mathbb{R}^{n+p}$.

Fact 6.1. *The map*

$$\iota_{n,p} : (\mathscr{OS}\mathrm{hape}^{n*}, d_{\mathscr{OS}^n}) \to (\mathscr{OS}\mathrm{hape}^{(n+p)*}, d_{\mathscr{OS}^{n+p}})$$

defined by $\iota_{n,p}([K]) = [\iota_{n,p}(K)]$ *is an isometry.*

Proof. Let K_1 and K_2 be convex bodies in $\mathbb{R}^n$ of unit intrinsic area. By property A6) of the intrinsic area, $\iota_{n,p}(K_1)$ and $\iota_{n,p}(K_2)$ have also unit intrinsic area. So it suffices to check that $V_2(K_1, K_2) = V_2(\iota_{n,p}(K_1), \iota_{n,p}(K_2))$, that is clear from 1.5 and A6). □

Let $\mathscr{OS}hape^{\infty *}$ be the union over n of $\mathscr{OS}hape^{n*}$, quotiented by the following equivalence relation: $[K_1]$ is equivalent to $[K_2]$ if and only if there exist $i, j \leq p$ such that $K_1 \subset \mathbb{R}^i$, $K_2 \subset \mathbb{R}^j$ and $[\iota_{i,p-i}(K_1)] = [\iota_{j,p-j}(K_2)]$. We will denote by $[K]_\infty$ an element of $\mathscr{OS}hape^{\infty *}$. For two representatives of $[K_1]_\infty, [K_2]_\infty \in \mathscr{OS}hape^{\infty *}$ in $\mathbb{R}^n$, let us define

$$d_{\mathscr{OS}^\infty}([K_1]_\infty, [K_2]_\infty) = d_{\mathscr{OS}^n}([K_1], [K_2]) \ .$$

It is easy to see that $d_{\mathscr{OS}^\infty}$ is well-defined and that it is actually a distance on $\mathscr{OS}hape^{\infty *}$.

The isometric embeddings $\iota_{n,p}$ induce isometric maps from $(\mathscr{S}hape^{n*}, d_{\mathscr{S}^n})$ to $(\mathscr{S}hape^{(n+p)*}, d_{\mathscr{S}^{n+p}})$, so in the same way we can define the set $\mathscr{S}hape^{\infty *}$ and the metric space $(\mathscr{S}hape^{\infty *}, d_{\mathscr{S}^\infty})$.

It follows from Theorems 1 and 3 that $(\mathscr{OS}hape^{\infty *}, d_{\mathscr{OS}^\infty})$ and $(\mathscr{S}hape^{\infty *}, d_{\mathscr{S}^\infty})$ are geodesic metric spaces. But it may happen that a sequence of convex bodies with non-empty interior in $\mathbb{R}^p$ converges to a convex body in $\mathscr{OS}hape^{\infty}$ when p goes to infinity, see below. This suggests that there may exist other shortest paths than the ones we know, and in consequence we don't know if those metric spaces have bounded curvature.

Fact 6.2. *Let* $K \in \mathcal{K}^{n*}$. *Let* $(\epsilon_p)_p$ *be a sequence of real numbers such that* $\sqrt{p}\epsilon_p \to 0$. *Then the sequence* $([\iota_{n,p}(K) + \epsilon_p B^{n+p}]_\infty)_p$ *converges in* $\mathscr{OS}\mathrm{hape}^{\infty *}$ *to* $[K]_\infty$.

Proof. We have $d_{\mathscr{OS}^\infty}([K]_\infty, [\iota_{n,p}(K)+\epsilon_p B^{n+p}]_\infty) = d_{\mathscr{OS}^{n+p}}([\iota_{n,p}(K)], [\iota_{n,p}(K)+\epsilon_p B^{n+p}])$, so

$$\cosh(d_{\mathscr{OS}^\infty}([K]_\infty, [\iota_{n,p}(K) + \epsilon_p B^{n+p}]_\infty)) = \frac{V_2(\iota_{n,p}(K), \iota_{n,p}(K) + \epsilon_p B^{n+p})}{\sqrt{V_2(\iota_{n,p}(K))V_2(\iota_{n,p}(K) + \epsilon_k B^{n+p})}} \ .$$

We have $V_2(\iota_{n,p}(K)) = V_2(K)$, and equations (1.2) and (1.7) give

$$V_2(\iota_{n,p}(K), \iota_{n,p}(K)+\epsilon_p B^{n+p}) = V_2(\iota_{n,p}(K))+\epsilon_p V_2(\iota_{n,p}(K), B^{n+p}) = V_2(K)+\frac{\epsilon_p}{2}V_1(B^{n+p-1})V_1(K)$$

and

$$\begin{aligned} V_2(\iota_{n,p}(K) + \epsilon_p B^{n+p}) &= V_2(\iota_{n,p}(K)) + 2\epsilon_p V_2(\iota_{n,p}(K), B^{n+p}) + \epsilon_p^2 V_2(B^{n+p}) \\ &= V_2(K) + \epsilon_p V_1(B^{n+p-1})V_1(K) + \epsilon_p^2(n+p-1)\pi \ . \end{aligned}$$

Equation (1.3) gives $V_1(B^{n+p-1}) \sim \sqrt{2\pi(n+p-1)}$, so $V_2(\iota_{n,p}(K), \iota_{n,p}(K) + \epsilon_p B^{n+p}) \to V_2(K)$ and $V_2(\iota_{n,p}(K)+\epsilon_p B^{n+p}) \to V_2(K)$, and this gives $d_{\mathscr{O}\mathscr{S}^\infty}([K]_\infty, [\iota_{n,p}(K)+\epsilon_p B^{n+p}]_\infty) \to 0$. $\square$

We show in the next proposition that the spaces $(\mathscr{O}\mathscr{S}hape^{\infty *}, d_{\mathscr{O}\mathscr{S}^\infty})$ and $(\mathscr{S}hape^{\infty *}, d_{\mathscr{S}^\infty})$ are not complete, hence not proper.

Proposition 6.3. *The space $(\mathscr{O}\mathscr{S}\mathrm{hape}^{\infty *}, d_{\mathscr{O}\mathscr{S}^\infty})$ (resp. $(\mathscr{S}\mathrm{hape}^{\infty *}, d_{\mathscr{S}^\infty})$) is not complete: the sequence of balls $([B^n]_\infty)_n$ (resp. $([\![B^n]\!]_\infty)_n$) is a diverging Cauchy sequence.*

Proof. Let $p \leq n$. Since the balls are invariant under isometries we have

$$d_{\mathscr{O}\mathscr{S}^\infty}([B^p]_\infty, [B^n]_\infty) = d_{\mathscr{S}^\infty}([\![B^p]\!]_\infty, [\![B^n]\!]_\infty) = d_{\mathscr{O}\mathscr{S}^n}([\iota_{p,n-p}(B^p)], [B^n]) \ ,$$

and by equations (1.2), (1.3) and (1.9) we have

$$\cosh(d_{\mathscr{O}\mathscr{S}^n}([\iota_{p,n-p}(B^p)], [B^n])) = \left(\frac{\sqrt{(n-1)}W_{n-1}}{\sqrt{(p-1)}W_{p-1}} \right) \ .$$

Since $W_n \sim \sqrt{\frac{\pi}{2n}}$, the sequence of balls (either in $\mathscr{O}\mathscr{S}hape^{\infty *}$ or $\mathscr{S}hape^{\infty *}$) are Cauchy sequences.

Moreover, suppose that $([B^n]_\infty)_n$ or $([\![B^n]\!]_\infty)_n$ converges to $[K]_\infty$ or $[\![K]\!]_\infty$ in $\mathscr{O}\mathscr{S}hape^{\infty *}$ or $\mathscr{S}hape^{\infty *}$. Since we have $d_{\mathscr{O}\mathscr{S}^\infty}([K]_\infty, [B^n]_\infty) = d_{\mathscr{S}^\infty}([\![K]\!]_\infty, [\![B^n]\!]_\infty)$, in any of these two cases we have that $([B^n]_\infty)_n$ converges to $[K]_\infty$ in $\mathscr{O}\mathscr{S}hape^{\infty *}$. And that is impossible.

Indeed, assume that K is a convex body of some Euclidean space $\mathbb{R}^p$. As B^n is invariant under the action of $O(n)$, then for any $\Phi \in O(p)$, by considering $O(p)$ as a subgroup of $O(n)$, we have that $([B^n]_\infty)_n$ converges to $\Phi[K]_\infty$. So K must be a ball, but from the computations above, $([B^n]_\infty)_n$ cannot converge to the class of a ball. $\square$

It would be interesting to explore more intensively the metric spaces $(\mathscr{O}\mathscr{S}hape^{\infty *}, d_{\mathscr{O}\mathscr{S}^\infty})$ and $(\mathscr{S}hape^{\infty *}, d_{\mathscr{S}^\infty})$, that is out of the scope of the present paper. The description of the completion of those spaces would certainly involve the study of "infinite dimensional convex bodies", as in [20, 13, 36, 37]. Note that in the case of $(\mathscr{O}\mathscr{S}hape^{\infty *}, d_{\mathscr{O}\mathscr{S}^\infty})$, it would be relevant to describe the space "$\mathbb{H}^\infty_\infty$" (that is, the inductive limit of the $\mathbb{H}^\infty_n$), and its completion.

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