

FUCHSIAN CONVEX BODIES: BASICS OF BRUNN–MINKOWSKI THEORY

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Abstract. The hyperbolic space \mathbb{H}^d can be defined as a pseudo-sphere in the $(d + 1)$ Minkowski space-time. In this paper, a Fuchsian group Γ is a group of linear isometries of the Minkowski space such that \mathbb{H}^d/Γ is a compact manifold. We introduce Fuchsian convex bodies, which are closed convex sets in Minkowski space, globally invariant for the action of a Fuchsian group. A volume can be associated to each Fuchsian convex body, and, if the group is fixed, Minkowski addition behaves well. Then Fuchsian convex bodies can be studied in the same manner as convex bodies of Euclidean space in the classical Brunn–Minkowski theory. For example, support functions can be defined, as functions on a compact hyperbolic manifold instead of the sphere. The main result is the convexity of the associated volume (it is log concave in the classical setting). This implies analogs of Alexandrov–Fenchel and Brunn–Minkowski inequalities. Here the inequalities are reversed.

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1 Introduction

There are two main motivations behind the definitions and results presented here. See next section for a precise definition of Fuchsian convex bodies, the main object of this paper, and Fuchsian convex surfaces (boundaries of Fuchsian convex bodies).

The first motivation is to show that the geometry of Fuchsian convex surfaces in the Minkowski space is the right analogue of the classical geometry of convex compact hypersurfaces in the Euclidean space. In the present paper, we show the analogue of the basics results of what is called Brunn–Minkowski theory. Roughly speaking, the matter is to study the relations between the sum and the volume of the bodies under consideration. Actually here we associate to each convex set the volume of another region of the space, determined by the convex set, so we will call it the *covolume* of the convex set. This generalization is as natural as, for example, going from the round sphere to compact hyperbolic surfaces. To strengthen this idea, existing results can be put into perspective. Indeed, Fuchsian convex surfaces are not new objects. As far I know, smooth Fuchsian hypersurfaces appeared in [OS83], see Section 3.3. The simplest examples of convex Fuchsian surfaces are convex hulls of the orbit of one point for the action of the Fuchsian group. They were considered in [NP91], in relation with the seminal papers [Pen87, EP88]. See also [CDM97]. The idea is to study hyperbolic problems via the extrinsic structure given by the Minkowski space. For a recent illustration see [EGM09]. The first study of Fuchsian surfaces for their own is probably [LS00]. The authors proved that for any Riemannian metric on a compact surface of genus ≥ 2 with negative curvature, there exists an isometric convex Fuchsian surface in the $2 + 1$ -Minkowski space, up to a quotient. In the Euclidean case, the analog problem is known as Weyl problem. A uniqueness result is also given. This kind of result about realization of abstract metrics by (hyper)surfaces invariant under a group action seems to go back to former papers of F. Labourie and to [Gro86]. The polyhedral analog of [LS00] is considered in [Fil11a]. An important intermediate result, about polyhedral infinitesimal rigidity in $d = 2$, was proved in [Sch07] (Fuchsian analogue of Dehn theorem). More recently, a Fuchsian analogue of the “Alexandrov prescribed curvature problem” in any dimension was proved in [Ber10], using optimal mass transport. The polyhedral $d = 2$ case was obtained in [Isk00]. In [Ber10] Fuchsian convex bodies are considered (under an equivalent definition) as analogue of convex bodies in the sense that support functions, radial functions and duality are introduced, similarly to the classical case. A solution for the Christoffel problem (prescribed sum of the radii of curvature in the regular case) for Fuchsian convex bodies should be given in [FV12] as well as for more general convex sets in the Minkowski space (with or without group action), similarly to [LS06].

The second motivation is that, up to a quotient, the results presented here are about the covolume defined by convex Cauchy surfaces in the simplest case of flat Lorentzian manifolds, namely the quotient of the interior of the future cone of the origin by a Fuchsian group. It is relevant to consider them in a larger class of flat

Lorentzian manifolds, known as maximal globally hyperbolic Cauchy-compact flat spacetimes. They were considered in the seminal paper [Mes07], see [ABBBGLSS07] and [Bar05, Bon05]. Roughly speaking, one could consider hypersurfaces in the Minkowski space invariant under a group of isometries whose set of linear isometries forms a Fuchsian group (translations are added). In $d = 2$, for such smooth strictly convex surfaces, a Minkowski theorem (generalizing Theorem 3.8 in this dimension) was proved recently in [BBZ11]. Maybe some of the basic objects introduced in the present paper could be extended to the point to these manifolds.

The paper is organized as follows. Section 2 introduces, among main definitions, the tool to study (Fuchsian) convex bodies, the support functions. The case of the C_+^2 Fuchsian convex bodies (roughly speaking, the ones with a sufficiently regular boundary) is treated in Section 3 and the one of polyhedral Fuchsian convex bodies in Section 4. These two sections are independent. In Section 5 the general results are obtained by polyhedral approximation. It appears that the proofs of the main results, even though very analogous to the classical ones, are simpler than in the Euclidean case.

2 Definitions

2.1 Fuchsian convex bodies. The Minkowski space-time of dimension $(d+1)$, $d \geq 1$, is \mathbb{R}^{d+1} endowed with the symmetric bilinear form

$$\langle x, y \rangle_- = x_1 y_1 + \cdots + x_d y_d - x_{d+1} y_{d+1}.$$

We will denote by \mathcal{F} the interior of the future cone of the origin. It is the set of future time-like vectors: the set of x such that $\langle x, x \rangle_- < 0$ (time-like) and the last coordinate of x for the standard basis is positive (future). The pseudo-sphere contained in \mathcal{F} at distance t from the origin of \mathbb{R}^{d+1} is

$$\mathbb{H}_t^d = \{x \in \mathbb{R}^{d+1} \mid \langle x, x \rangle_- = -t^2, x_{d+1} > 0\}.$$

All along the paper we identify \mathbb{H}_1^d with the hyperbolic space \mathbb{H}^d . In particular the isometries of \mathbb{H}^d are identified with the linear isometries of the Minkowski space keeping \mathbb{H}_1^d invariant [BP92, A.2.4]. Note that for any point $x \in \mathcal{F}$, there exists t such that $x \in \mathbb{H}_t^d$.

DEFINITION 2.1. A *Fuchsian group* Γ is a subgroup of the linear isometries group of \mathbb{R}^{d+1} , fixing setwise \mathcal{F} and acting freely cocompactly on \mathbb{H}^d (i.e. \mathbb{H}^d/Γ is a compact manifold).

A *Fuchsian convex body* is the data of a convex closed proper subset K of \mathcal{F} , together with a Fuchsian group Γ , such that $\Gamma K = K$. A Γ -convex body is a Fuchsian convex body with Fuchsian group Γ .

A *Fuchsian convex surface* is the boundary of a Fuchsian convex body.

A Fuchsian convex body has to be thought as the analogue of a convex body (compact convex set), with the compactness condition replaced by a “cocompactness” condition (we will see that a Fuchsian convex body is never bounded). Joan Porti pointed out to the author that what is done in this paper is probably true without the requirement that the group has no torsion.

We will adapt the classical theory to the Fuchsian case. For that one we mainly follow [Sch93].

Examples. The simplest examples of Fuchsian convex surfaces are the \mathbb{H}_t^d (note that all Fuchsian groups act freely and cocompactly on \mathbb{H}_t^d). Their convex sides are Fuchsian convex bodies, denoted by B_t^d , and B_1^d is sometimes denoted by B^d or B . This example shows that a given convex set can be a Fuchsian convex body for many Fuchsian groups.

Given a Fuchsian group Γ , we will see in the remaining of the paper two ways of constructing convex Fuchsian bodies. First, given a finite number of points in \mathcal{F} , the convex hull of their orbits for Γ is a Fuchsian convex body, see Section 4.1, where a dual construction is introduced. Second, we will see in Section 3.1 that any function on the compact hyperbolic manifold \mathbb{H}^d/Γ satisfying a differential relation corresponds to a Fuchsian convex body. Hence the question of examples reduces to the question of finding the group Γ , that implies to find compact hyperbolic manifolds. Standard concrete examples of compact hyperbolic manifolds can be easily found in the literature about hyperbolic manifolds. For a general construction in any dimension see [GP88].

Notwithstanding it is not obvious to get explicit generators. Of course the case $d = 1$ is totally trivial as a Fuchsian group is generated by a boost $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, for a non-zero real t . For $d = 2$, explicit generators can be constructed following [Mas01]. For Figure 2 and a computation at the end of the paper (the figure comes from a part of a Fuchsian convex body that can be manipulate on the author’s webpage), the group is the simplest acting on \mathbb{H}^2 , namely the one having a regular octagon as fundamental domain in a disc model. Generators are given in [Kat92].

Remark on the signature of the bilinear form. The classical theory of convex bodies uses the usual scalar product on \mathbb{R}^{d+1} . Here we used the usual bilinear form of signature $(d, 1)$. A natural question is to ask what happens if we consider a bilinear form of signature $(d + 1 - k, k)$. (Obviously, the vector structure, the volume, the Levi-Civita connection (and hence the geodesics), the topology and the notion of convexity don’t depend on the signature. Moreover, any linear map preserving the bilinear form is of determinant one, hence preserves the volume.)

Let us consider first the case of the usual bilinear form with signature $(d - 1, 2)$ ($d \geq 3$). The set of vectors of pseudo-norm -1 is a model of the Anti-de Sitter space, which is the Lorentzian analogue of the Hyperbolic space. First of all, we need groups of linear isometries acting cocompactly on the Anti-de Sitter space. They exist only

in odd dimensions [BZ04]. Moreover, Anti-de Sitter space does not bound a convex set.

Finally, another interest of the present construction is that, as noted in the introduction, some objects introduced here could serve to study some kind of flat Lorentzian manifolds (with compact Cauchy surface), which can themselves be related to some problems coming from General Relativity. It is not clear if as many attention is given to pseudo-Riemannian manifolds with different signatures.

2.2 Support planes. For a subset A of \mathbb{R}^{d+1} , a *support plane* of A at x is an hyperplane \mathcal{H} with $x \in A \cap \mathcal{H}$ and A entirely contained in one side of \mathcal{H} .

LEMMA 2.2. *Let K be a Γ -convex body. Then*

- (i) K is not contained in a codimension > 0 plane.
- (ii) K is future convex
 - (a) through each boundary point there is a support plane;
 - (b) all support planes are space-like;
 - (c) K is contained in the future side of its support planes.

Proof. By definition K is not empty. Let $x \in K$. As $K \subset \mathcal{F}$, there exists a t such that $x \in \mathbb{H}_t^d$, and by definition, all the elements of the orbit Γx of x belong to $K \cap \mathbb{H}_t^d$. Suppose that K is contained in a codimension > 0 hyperplane \mathcal{H} . Then there would exist a codimension 1 hyperplane \mathcal{H}' with $\mathcal{H} \subset \mathcal{H}'$, and $\Gamma x \in \mathcal{H}' \cap \mathbb{H}_t^d$. This means that on \mathbb{H}_t^d (which is homothetic to the hyperbolic space for the induced metric), Γx is contained in a totally geodesic hyperplane, a hypersphere or a horosphere (depending on \mathcal{H}' to be time-like, space-like or light-like), that is clearly impossible. (i) is proved.

(ii)(a) is a general property of convex closed subset of \mathbb{R}^{d+1} [Sch93, 1.3.2].

Let $x \in K$ and let \mathcal{H} be the support plane of K at x . There exists t such that $\Gamma x \subset \mathbb{H}_t^d$, and all elements of Γx must be on one side of $\mathcal{H} \cap \mathbb{H}_t^d$ on \mathbb{H}_t^d . Clearly $\mathcal{H} \cap \mathbb{H}_t^d$ can't be a totally geodesic hyperplane (of \mathbb{H}_t^d), and it can't either be a horosphere by Sublemma 2.3. Hence \mathcal{H} must be space-like, that gives (ii)(b). The fact that all elements of Γx belong to \mathbb{H}_t^d implies that K is in the future side of its support planes, hence (ii)(c). \square

SUBLEMMA 2.3. *Let Γ be a group of isometries acting cocompactly on the hyperbolic space \mathbb{H}^d . For any $x \in \mathbb{H}^d$, the orbit Γx meets the interior of any horoball.*

Proof. As the action of Γ on \mathbb{H}^d is cocompact, it is well-known that the orbit Γx is discrete and that the Dirichlet regions for Γx

$$D_a(\Gamma) = \{p \in \mathbb{H}^d \mid d(a, p) \leq d(\gamma a, p), \forall \gamma \in \Gamma \setminus \{Id\}\}, a \in \Gamma x \quad (1)$$

where d is the hyperbolic distance, are bounded [Rat06]. The sublemma is a characteristic property of discrete sets with bounded Dirichlet regions [CDM97, Lemma 3]. \square

LEMMA 2.4. *Let K be a Γ -convex body and $x \in K$. For any $\lambda \geq 1$, $\lambda x \in K$.*

Proof. From the definition of K , it is not hard to see that it has non empty interior. And as K is closed, if the lemma was false, there would exist a point on the boundary of K and a support plane at this point such that x in its past, that is impossible because of Lemma 2.2. \square

Let us recall the following elementary results, see e.g. [Rat06, 3.1.1, 3.1.2].

SUBLEMMA 2.5. (i) *If x and y are nonzero non space-like vectors in \mathbb{R}^{d+1} , both past or future, then $\langle x, y \rangle_- \leq 0$ with equality if and only if x and y are linearly dependent light-like vectors.*

(ii) *If x and y are nonzero non space-like vectors in \mathbb{R}^{d+1} , both past (resp. future), then the vector $x + y$ is past (resp. future) non space-like. Moreover $x + y$ is light-like if and only if x and y are linearly dependent light-like vectors.*

A future time-like vector η orthogonal to a support plane at x of a future convex set A is called an *inward normal* of A at x . This means that $\forall y \in A$, $y - x$ and η are two future time-like vectors at the point x , then by Sublemma 2.5,

$$\forall y \in A, \langle \eta, y - x \rangle_- \leq 0, \text{ i.e. } \langle \eta, y \rangle_- \leq \langle \eta, x \rangle_-$$

or equivalently the sup on all $y \in A$ of $\langle \eta, y \rangle_-$ is attained at x . Notice that the set

$$\{y \in \mathbb{R}^{d+1} | \langle y, \eta \rangle_- = \langle x, \eta \rangle_-\}$$

is the support hyperplane of A at x with inward normal η .

LEMMA 2.6. *Let K be a Γ -convex body. For any future time-like vector η , $\sup\{\langle x, \eta \rangle_- | x \in K\}$ exists, is attained at a point of K and is negative. In particular any future time-like vector η is an inward normal of K .*

A future time-like vector η is the inward normal of a single support hyperplane of K .

Proof. From (i) of Sublemma 2.5, $\{\langle x, \eta \rangle_- | x \in K\}$ is bounded from above by zero hence the sup exists. The sup is a negative number, as a sufficiently small translation of the vector hyperplane \mathcal{H} orthogonal to η in direction of \mathcal{F} does not meet K . This follows from the separation theorem [Sch93, 1.3.4], because the origin is the only common point between \mathcal{H} and the boundary of \mathcal{F} . As K is closed, the sup is attained when the parallel displacement of \mathcal{H} meets K .

Suppose that two different support hyperplanes of K have the same inward normal. Hence one is contained in the past of the other, that is impossible. \square

2.3 Support functions. Let K be a Γ -convex body. The *extended support function* H of K is

$$\forall \eta \in \mathcal{F}, H(\eta) = \sup\{\langle x, \eta \rangle_- | x \in K\}. \quad (2)$$

We know from Lemma 2.6 that it is a negative function on \mathcal{F} . As an example the extended support function of B_t^d is equal to $-t\sqrt{-\langle \eta, \eta \rangle_-}$.

DEFINITION 2.7. A function $f : A \rightarrow \mathbb{R}$ on a convex subset A of \mathbb{R}^{d+1} is sublinear (on A) if it is positively homogeneous of degree one:

$$\forall \eta \in A, f(\lambda\eta) = \lambda f(\eta) \quad \forall \lambda > 0, \quad (3)$$

and subadditive:

$$\forall \eta, \mu \in A, f(\eta + \mu) \leq f(\eta) + f(\mu). \quad (4)$$

A sublinear function is convex, in particular it is continuous (by assumptions it takes only finite values in A). (It is useful to note that for a positively homogeneous of degree one function, convexity and sublinearity are equivalent.) It is straightforward from the definition that an extended support function is sublinear and Γ -invariant. It is useful to expand the definition of extended support function to the whole space. The *total support function* of a Γ -convex body K is

$$\forall \eta \in \mathbb{R}^{d+1}, \tilde{H}(\eta) = \sup\{\langle x, \eta \rangle_- \mid x \in K\}. \quad (5)$$

We will consider the total support function for any convex subset of \mathbb{R}^{d+1} . The infinite value is allowed. We have the following important property, see [Hör07, Theorem 2.2.8].

PROPOSITION 2.8. Let f be a lower semi-continuous, convex and positively homogeneous of degree one function on \mathbb{R}^{d+1} (the infinite value is allowed). The set

$$F = \{x \in \mathbb{R}^{d+1} \mid \langle x, \eta \rangle_- \leq f(\eta) \quad \forall \eta \in \mathbb{R}^{d+1}\}$$

is a closed convex set with total support function f .

From the definition we get:

LEMMA 2.9. A convex subset of \mathbb{R}^{d+1} is a point if and only if its total support function is a linear form. (If the point is p , the linear form is $\langle \cdot, p \rangle_-$.)

In particular, the total support function of a Fuchsian convex body is never a linear form.

The relation between the extended support function and the total support function is as follows.

LEMMA 2.10. The total support function \tilde{H} of a Γ -convex body with extended support function H is equal to:

- H on \mathcal{F} ,
- 0 on the future light-like vectors and at 0,
- $+\infty$ elsewhere.

Moreover \tilde{H} is a Γ invariant sublinear function.

Proof. We have the following cases:

- If η is future time-like then $\tilde{H}(\eta) = H(\eta)$.
- If η is past time-like or past light-like, then by (i) of Sublemma 2.5 for $x \in K$, $\langle x, \eta \rangle_- > 0$, and by Lemma 2.4, $\tilde{H}(\eta) = +\infty$.
- If η is space-like, as seen in the proof of (ii) of Lemma 2.2, there exists points of K on both side of the orthogonal (for $\langle \cdot, \cdot \rangle_-$) of η . Hence there exists $x \in K$ with $\langle x, \eta \rangle_- > 0$, and by the preceding argument, $\tilde{H}(\eta) = +\infty$.
- If η is future light-like, then $\tilde{H}(\eta) = 0$. As \tilde{H} is lower semi-continuous (as supremum of a family of continuous functions) and as $\tilde{H} = +\infty$ outside of the future cone, this follows from Sublemma 2.11.
- By definition, $\tilde{H}(0) = 0$.

That \tilde{H} is a Γ invariant sublinear function follows easily. \square

SUBLEMMA 2.11. *Let H be a sublinear function on \mathcal{F} with finite values. Let us extend it as a convex function on \mathbb{R}^{d+1} by giving the value $+\infty$ outside \mathcal{F} . Let \tilde{H} be the lower semi-continuous hull of H : $\tilde{H}(x) = \liminf_{x \rightarrow y} H(y)$.*

If H is invariant under the action of Γ , then H is negative or $H \equiv 0$ on \mathcal{F} , and $\tilde{H} = 0$ on $\partial\mathcal{F}$.

Note that $H \equiv 0$ is the support function of (the closure of) \mathcal{F} .

Proof. By cocompactness, there exists positive c and d such that

$$-c\sqrt{-\langle \eta, \eta \rangle_-} \leq H(\eta) \leq -d\sqrt{-\langle \eta, \eta \rangle_-}.$$

This proves that $\tilde{H} = 0$ on $\partial\mathcal{F}$ as for any $\ell \in \partial\mathcal{F}^*$ and any $\eta \in \mathcal{F}$, $\tilde{H}(\ell) = \lim_{t \downarrow 0} H(\ell + t(x - \ell))$ (see for example Theorem 7.5 in [Roc97]). In the same way we get that $\tilde{H}(0) = 0$.

As \tilde{H} is convex and equal to 0 on $\partial\mathcal{F}$, it is non-positive on \mathcal{F} . Suppose that there exists $x \in \mathcal{F}$ with $\tilde{H}(x) = 0$, and let $y \in \mathcal{F} \setminus \{x\}$. By homogeneity, $\tilde{H}(\lambda x) = 0$ for all $\lambda > 0$. Up to choose an appropriate λ , we can suppose that the line joining x and y meets $\partial\mathcal{F}$ in two points. Let ℓ be the one such that there exists $\lambda \in]0, 1[$ such that $x = \lambda\ell + (1 - \lambda)y$. By convexity and because $\tilde{H}(x) = \tilde{H}(\ell) = 0$, we get $0 \leq \tilde{H}(y)$, hence $\tilde{H}(y) = 0$. \square

LEMMA 2.12. *Let H be a negative sublinear Γ -invariant function on \mathcal{F} . The set*

$$K = \{x \in \mathcal{F} \mid \langle x, \eta \rangle_- \leq H(\eta) \forall \eta \in \mathcal{F}\}$$

is a Γ -convex body with extended support function H .

Proof. Let \tilde{H} be as in Sublemma 2.11. From Proposition 2.8, the set

$$\tilde{K} = \{x \in \mathbb{R}^{d+1} \mid \langle x, \eta \rangle_- \leq \tilde{H}(\eta) \forall \eta \in \mathbb{R}^{d+1}\}$$

is a closed convex set, with total support function \tilde{H} . Let us see that $\tilde{K} = K$.

As $\tilde{H}(\eta) = +\infty$ outside the closure $\overline{\mathcal{F}}$ of the future cone we have

$$\tilde{K} = \{x \in \mathbb{R}^{d+1} \mid \langle x, \eta \rangle_- \leq \tilde{H}(\eta) \forall \eta \in \overline{\mathcal{F}}\}.$$

For $\eta \in \mathcal{F}$, $\tilde{H}(\eta) \leq 0$, it follows that \tilde{K} is contained in $\overline{\mathcal{F}}$:

$$\tilde{K} = \{x \in \overline{\mathcal{F}} \mid \langle x, \eta \rangle_- \leq \tilde{H}(\eta) \forall \eta \in \overline{\mathcal{F}}\}.$$

As H is Γ -invariant, \tilde{H} and \tilde{K} are Γ -invariant too. For $x \in \tilde{K} \cap \partial\mathcal{F}$, the origin is an accumulating point of Γx from Sublemma 2.13. So for any $\eta \in \mathcal{F}$, $\tilde{H}(\eta)$, which is the sup of $\langle x, \eta \rangle_-$ for $x \in \tilde{K}$, should be zero, that is false. Hence

$$\tilde{K} = \{x \in \mathcal{F} \mid \langle x, \eta \rangle_- \leq \tilde{H}(\eta) \forall \eta \in \overline{\mathcal{F}}\}$$

and as $\tilde{H}(\eta) = 0$ on $\partial\mathcal{F}$ we get

$$\tilde{K} = \{x \in \mathcal{F} \mid \langle x, \eta \rangle_- \leq \tilde{H}(\eta) \forall \eta \in \mathcal{F}\} = K.$$

The remainder is easy. □

SUBLEMMA 2.13. *Let Γ be a Fuchsian group and let x be a future light-like vector. Then the origin is an accumulating point of Γx .*

Proof. Suppose it is false. As Γ acts cocompactly on \mathbb{H}^d , there exists an horizontal space-like hyperplane S such that a fundamental domain on \mathbb{H}^d for the action of Γ lies below S . If the origin is not an accumulating point, then there exists $\lambda > 0$ such that the horoball

$$\{y \in \mathbb{H}^d \mid -1 \leq \langle \lambda x, y \rangle_- < 0\}$$

and its images for the action of Γ remain above S . This contradicts the definition of fundamental domain. □

The *polar dual* K^* of a Γ -convex body K is, if H is the extended support function of K :

$$K^* = \{x \in \mathcal{F} \mid H(x) \leq -1\}.$$

For example, $(B_t^d)^* = B_{1/t}^d$. It is not hard to see that K^* is a Γ -convex body, and that $K^{**} = K$ (see the convex bodies case [Sch93, 1.6.1]). Moreover the points of the boundary of K^* are the $\frac{-1}{H(\eta)}\eta$ for $\eta \in \mathbb{H}^d$. The inverse of this map is the projection $f(x) = \frac{x}{\sqrt{-\langle x, x \rangle_-}}$. Hence, exchanging the roles of K and K^* , we get that the projection of a Fuchsian convex body along rays from the origin gives is a homeomorphism between ∂K and \mathbb{H}^d .

2.4 Minkowski sum and covolume. The (*Minkowski*) addition of two sets $A, B \subset \mathbb{R}^{d+1}$ is defined as

$$A + B := \{a + b \mid a \in A, b \in B\}.$$

It is well-known that the addition of two convex sets is a convex set. Moreover the sum of two future time-like vectors is a future time-like vector, in particular it is never zero. So the sum of two Γ -convex bodies is contained in \mathcal{F} and closed [RW98, 3.12]. As a Fuchsian group Γ acts by linear isometries, the sum is a Γ -convex body, and the space $\mathcal{K}(\Gamma)$ of Γ -convex bodies is invariant under the addition. Note also that $\mathcal{K}(\Gamma)$ is invariant under multiplication by positive scalars. It is straightforward to check that extended support functions behave well under these operations:

$$H_{K+L} = H_K + H_L, \quad K, L \in \mathcal{K}(\Gamma),$$

$$H_{\lambda K} = \lambda H_K, \quad \lambda > 0, K \in \mathcal{K}(\Gamma).$$

Note also that from the definition of the extended support function,

$$K \subset L \Leftrightarrow H_K \leq H_L.$$

Identifying Γ -convex bodies with their support functions, $\mathcal{K}(\Gamma)$ is a cone in the vector space of homogeneous of degree 1, continuous, real, Γ -invariant, functions on \mathcal{F} . By homogeneity this corresponds to a cone in the vector space of continuous real Γ -invariant functions on $\mathbb{H}^d \subset \mathcal{F}$, and to a cone in the vector space of continuous real functions on the compact hyperbolic manifold \mathbb{H}^d/Γ . A function in one of this two last cones is called a *support function*.

Let $K \in \mathcal{K}(\Gamma)$. Its *covolume* $\text{covol}(K)$ is the volume of $(\mathcal{F} \setminus K)/\Gamma$ (for the Lebesgue measure of \mathbb{R}^{d+1}). It is a finite positive number and

$$\text{covol}(\lambda K) = \lambda^{d+1} \text{covol}(K).$$

Note that

$$K \subset L \Rightarrow \text{covol}(K) \geq \text{covol}(L).$$

As defined above, the covolume of a Γ -convex body K is the volume of a compact set of \mathbb{R}^{d+1} , namely the volume of the intersection of $\mathcal{F} \setminus K$ with a fundamental cone for the action of Γ . For such compact (non-convex) sets there is a Brunn–Minkowski theory, see for example [Gar02]. See also [BE99]. But this does not give results about covolume of Γ -convex bodies. The reason is that, for two Γ -convex bodies K_1 and K_2 , $\mathcal{F} \setminus (K_1 + K_2)$ (from which we define the covolume of $K_1 + K_2$) is not equal to $(\mathcal{F} \setminus K_1) + (\mathcal{F} \setminus K_2)$. For example in $d = 1$, $\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 5/8 \\ 9/8 \end{pmatrix} \in (\mathcal{F} \setminus B + \mathcal{F} \setminus B)$ but does not belong to $\mathcal{F} \setminus (B + B)$.

3 C_+^2 Case

The first subsection is an adaptation of the classical case [Sch93]. The remainder is the analog of [Ale38] (in [Ale96]). See also [BF87, Lei93, Hör07, Bus08, and GMTZ10] for a kind of extension.

The objects and results in this section which can be defined intrinsically on a hyperbolic manifold are already known in more generality, see [OS83] and the references therein. See also Section 3.3.

3.1 Regularity of the support function

Differentiability. Let K be a Γ -convex body with extended support function H , and let $\eta \in \mathcal{F}$. From Lemma 2.6 there exists a unique support hyperplane \mathcal{H} of K with inward normal η .

LEMMA 3.1. *The intersection F of \mathcal{H} and K is reduced to a single point p if and only if H is differentiable at $\eta \in \mathcal{F}$. In this case $p = \nabla_\eta H$ (the gradient for $\langle \cdot, \cdot \rangle_-$ of H at η).*

Proof. As H is convex all one-sided directional derivatives exist [Sch93, p. 25]. Let us denote such derivative in the direction of $u \in \mathbb{R}^{d+1}$ at the point η by $d_\eta H(u)$. The proof of the lemma is based on the following fact:

The function $\mathbb{R}^{d+1} \ni u \mapsto d_\eta H(u)$ is the total support function of F .

Indeed, if H is differentiable at η , the fact says that the total support function of F is a linear form, and from Lemma 2.9, F is a point. Conversely, if F is a point p , from Lemma 2.9 its total support function is a linear form, hence partial derivatives of H exist and as H is convex, this implies differentiability [Sch93, 1.5.6]. Moreover for all $u \in \mathbb{R}^{d+1}$, $\langle p, u \rangle_- = d_\eta H(u)$.

Now we prove the fact. The function $d_\eta H$ is sublinear on \mathbb{R}^{d+1} [Sch93, 1.5.4], Proposition 2.8 applies and $d_\eta H$ is the total support function of

$$F' = \{x \in \mathbb{R}^{d+1} \mid \langle x, u \rangle_- \leq d_\eta H(u) \forall u \in \mathbb{R}^{d+1}\}.$$

We have to prove that $F' = F$. Let \tilde{H} be the extension of H to \mathbb{R}^{d+1} . By definition of directional derivative, the sublinearity of \tilde{H} gives $d_\eta H \leq \tilde{H}$. From the proof of Lemma 2.12, this implies that $F' \subset K$. In particular, for $y \in F'$, $\langle y, \eta \rangle_- \leq H(\eta)$. On the other hand $y \in F'$ implies $\langle y, -\eta \rangle_- \leq d_\eta H(-\eta) = -H(\eta)$ (the last equality follows from the definition of directional derivative, using the homogeneity of H). Then $\langle y, \eta \rangle_- = H(\eta)$ so $y \in \mathcal{H}$, hence $F' \subset F = \mathcal{H} \cap K$.

Let $y \in F$. By definition $\langle y, \eta \rangle_- = H(\eta)$ and for any $w \in \mathcal{F}$, $\langle y, w \rangle_- \leq H(w)$. For sufficiently small positive λ and any $u \in \mathbb{R}^{d+1}$, $w = \eta + \lambda u$ is future time-like and

$$\langle y, u \rangle_- \leq \frac{H(\eta + \lambda u) - H(\eta)}{\lambda}$$

so when $\lambda \rightarrow 0$ we have $\langle y, u \rangle_- \leq d_\eta H(u)$ hence $F \subset F'$. The fact is proved. \square

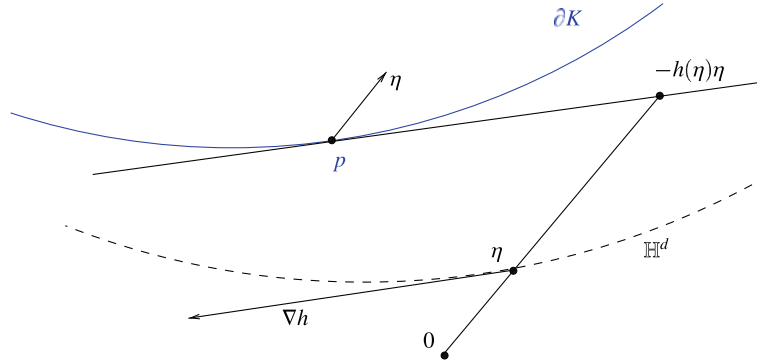


Figure 1: Recovering the convex body from its support function in the Minkowski space

If the extended support function H of a Γ -convex body K is differentiable, the above lemma allows to define the map

$$\tilde{G}(\eta) = \nabla_{\eta} H$$

from \mathcal{F} to $\partial K \subset \mathbb{R}^{d+1}$. This can be expressed in term of h , the restriction of H to \mathbb{H}^d . We use “hyperbolic coordinates” on \mathcal{F} : an orthonormal frame on \mathbb{H}^d extended to an orthonormal frame of \mathcal{F} with the decomposition $r^2 g_{\mathbb{H}^d} - dr^2$ of the metric on \mathcal{F} . $\nabla_{\eta} H$ has $d + 1$ entries, and, at $\eta \in \mathbb{H}^d$, the d first ones are the coordinates of $\nabla_{\eta} h$ (here ∇ is the gradient on \mathbb{H}^d). We identify $\nabla_{\eta} h \in T_{\eta} \mathbb{H}^d \subset \mathbb{R}^{d+1}$ with a vector of \mathbb{R}^{d+1} . The last component of $\nabla_{\eta} H$ is $-\partial H / \partial r(\eta)$, and, using the homogeneity of H , it is equal to $-h(\eta)$ when $\eta \in \mathbb{H}^d$. Note that at such a point, $T_{\eta} \mathcal{F}$ is the direct sum of $T_{\eta} \mathbb{H}^d$ and η . It follows that, for $\eta \in \mathbb{H}^d$,

$$\nabla_{\eta} H = \nabla_{\eta} h - h(\eta)\eta. \tag{6}$$

This has a clear geometric interpretation, see Figure 1.

C^2 support function If the extended support function H is C^2 , \tilde{G} is C^1 , and its differential \tilde{W} satisfies

$$\langle \tilde{W}_{\eta}(X), Y \rangle_{-} = D_{\eta}^2 H(X, Y).$$

We denote by G the restriction of \tilde{G} to \mathbb{H}^d and by W its differential (the *reversed shape operator*). If T_{ν} is the hyperplane of \mathbb{R}^{d+1} orthogonal to $\nu \in \mathbb{H}^d$ for $\langle \cdot, \cdot \rangle_{-}$, W is considered as a map from T_{ν} to T_{ν} . We get from (6), or from the equation above, the Gauss formula and the 1-homogeneity of H , using again hyperbolic coordinates on \mathcal{F} :

$$W_{ij} = (\nabla^2 h)_{ij} - h\delta_{ij}, \tag{7}$$

with ∇^2 the second covariant derivative (the Hessian) on \mathbb{H}^d , δ_{ij} the Kronecker symbol and h the restriction of H to \mathbb{H}^d . In particular W is symmetric, and its real

eigenvalues r_1, \dots, r_d are the *radii of curvature* of K . Taking the trace on both parts of the equation above leads to

$$r_1 + \dots + r_d = \Delta_{\mathbb{H}^d} h - dh \tag{8}$$

where $\Delta_{\mathbb{H}^d}$ is the Laplacian on the hyperbolic space. It is easy to check that, for $\gamma \in \Gamma$, $\nabla_{\gamma\eta} H = \gamma \nabla_{\eta} H$ and $D_{\gamma\eta}^2 H = D_{\eta}^2 H$. In particular the objects introduced above can be defined on \mathbb{H}^d/Γ .

C_+^2 body. Let K be a Γ -convex body. The *Gauss map* N is a multivalued map which associates to each x in the boundary of K the set of unit inward normals of K at x , which are considered as elements of \mathbb{H}^d . If the boundary of K is a C^2 hypersurface and if the Gauss map is a C^1 -homeomorphism from the boundary of K to \mathbb{H}^d , K is C_+^2 . In this case we can define the *shape operator* $B = \nabla N$, which is a self-adjoint operator. Its eigenvalues are the *principal curvatures* κ_i of K , and they are never zero as B has maximal rank by assumption. As K is convex, it is well-known that its principal curvatures are non-negative, hence they are positive. (This implies that K is actually strictly convex.)

LEMMA 3.2. *Under the identification of a Γ -convex body with its support function, the set of C_+^2 Γ -convex body is $C_+^2(\Gamma)$, the set of negative C^2 functions h on $M = \mathbb{H}^d/\Gamma$ such that*

$$((\nabla^2 h)_{ij} - h\delta_{ij}) > 0 \tag{9}$$

(positive definite) for any orthonormal frame on M .

It follows that in the C_+^2 case $G = N^{-1}$, $W = B^{-1}$, and $r_i = \frac{1}{\kappa_i \circ N^{-1}}$.

Proof. Let K be a C_+^2 Γ -convex body, h its support function and H its extended support function (h is the restriction of H to \mathbb{H}^d). For any $\eta \in \mathbb{H}^d$ we have

$$h(\eta) = \langle N^{-1}(\eta), \eta \rangle_-, \tag{10}$$

and for $\eta \in \mathcal{F}$, introducing the 0-homogeneous extension \tilde{N}^{-1} of N^{-1} we obtain

$$D_{\eta} H(X) = \langle \tilde{N}^{-1}(\eta), X \rangle_- + \langle D_{\eta} \tilde{N}^{-1}(X), \eta \rangle_-,$$

but $D_{\eta} \tilde{N}^{-1}(X)$ belongs to the support hyperplane of K with inward normal η so $D_{\eta} H(X) = \langle \tilde{N}^{-1}(\eta), X \rangle_-$. Hence $D_{\eta}^2 H(X, Y) = \langle B^{-1}(X), Y \rangle_-$, in particular H is C^2 , so h is C^2 and (9) is known. As h is Γ -invariant, we get a function of $C_+^2(\Gamma)$.

Now let $h \in C_+^2(\Gamma)$. We also denote by h the Γ -invariant map on \mathbb{H}^d which projects on h , and by H the 1-homogeneous extension of h to \mathcal{F} . The 1-homogeneity and (9) imply that H is convex (in the hyperbolic coordinates, row and column of the Hessian of H corresponding to the radial direction r are zero), hence negative sublinear Γ -invariant, so it is the support function of a Γ -convex body K by Lemma 2.12. As h is C^2 , we get a map G from \mathbb{H}^d to $\partial K \subset \mathbb{R}^{d+1}$ which is C^1 , and

regular from (7) and (9). Moreover G is surjective by Lemma 2.6. It follows that ∂K is C^1 . This implies that each point of ∂K has a unique support plane [Sch93, p. 104], i.e that the map G is injective. Finally it is a C^1 homeomorphism.

Let K^* be the polar dual of K . We know that the points on the boundary of K^* are graphs above \mathbb{H}^d as they have the form $\eta/(-h(\eta))$ for $\eta \in \mathbb{H}^d$. Hence ∂K^* is C^2 as h is. Moreover the Gauss map image of the point $\eta/(-h(\eta))$ of ∂K^* is $G(\eta)/\sqrt{-\langle G(\eta), G(\eta) \rangle_-}$: the Gauss map of K^* is a C^1 homeomorphism. It follows that K^* is C^2_+ . In particular its support function is C^2 . Repeating the argument, it follows that the boundary of $K^{**} = K$ is C^2 . \square

To simplify the matter in the following, we will restrict ourselves to smooth (C^∞) support functions, although this restriction will be relevant only in Section 3.4. We denote by $C^\infty_+(\Gamma)$ the subset of smooth elements of $C^2_+(\Gamma)$. It corresponds to C^∞_+ Γ -convex bodies, i.e. Γ -convex bodies with smooth boundary and with the Gauss map a C^1 diffeomorphism (hence smooth).

LEMMA 3.3. $C^\infty_+(\Gamma)$ is a convex cone and

$$C^\infty_+(\Gamma) - C^\infty_+(\Gamma) = C^\infty(\Gamma)$$

(any smooth function on \mathbb{H}^d/Γ is the difference of two functions of $C^\infty_+(\Gamma)$).

Proof. It is clear that $C^\infty_+(\Gamma)$ is a convex cone. Let $h_1 \in C^\infty_+(\Gamma)$ and $Z \in C^\infty(\Gamma)$. As \mathbb{H}^d/Γ is compact, for t sufficiently large, $Z + th_1$ satisfies (9) and is a negative function, hence there exists $h_2 \in C^\infty_+(\Gamma)$ such that $Z + th_1 = h_2$. \square

3.2 Covolume and Gaussian curvature operator. Let K be a C^2_+ Γ -convex body and let $P(K)$ be \mathcal{F} minus the interior of K . As $P(K)/\Gamma$ is compact, the divergence theorem gives

$$\int_{P(K)/\Gamma} \operatorname{div} X \, dP(K) = - \int_{\partial K/\Gamma} \langle X, \eta \rangle_- \, d\partial K,$$

where η is the unit outward normal of $\partial K/\Gamma$ in $P(K)/\Gamma$ (hence it corresponds in the universal cover to the unit inward normal of K). If X is the position vector in \mathcal{F} we get

$$(d + 1)\operatorname{covol}(K) = - \int_{\partial K/\Gamma} h \circ N \, d\partial K$$

with h the support function of K and N the Gauss map.

The *Gaussian curvature* (or Gauss–Kronecker curvature) κ of K is the product of the principal curvatures. We will consider the map κ^{-1} which associates to each $h \in C^\infty_+(\Gamma)$ the inverse of the Gaussian curvature of the convex body supported by h :

$$\kappa^{-1}(h) = \prod_{i=1}^d r_i(h) \stackrel{(7)}{=} \det((\nabla^2 h)_{ij} - h\delta_{ij}). \tag{11}$$

As the curvature is the Jacobian of the Gauss map, we get

$$(d+1)\text{covol}(K) = - \int_M h\kappa^{-1}(h) \, dM$$

where dM is the volume form on $M = \mathbb{H}^d/\Gamma$. Finally let us consider the covolume as a functional on $C_+^\infty(\Gamma)$, which extension to the whole $C^\infty(\Gamma)$ is immediate:

$$\text{covol}(X) = -\frac{1}{d+1} (X, \kappa^{-1}(X)), \quad X \in C^\infty(\Gamma) \quad (12)$$

with (\cdot, \cdot) the scalar product on $L^2(M)$.

We will consider $C^\infty(\Gamma)$ as a Fréchet space with the usual seminorms

$$\|f\|_n = \sum_{i=1}^n \sup_{x \in M} |\nabla^i f(x)|,$$

with ∇^i the i th covariant derivative and $|\cdot|$ the norm, both given by the Riemannian metric of M . All derivatives will be directional (or Gâteaux) derivatives in Fréchet spaces as in [Ham82]:

$$D_Y \text{covol}(X) = \lim_{t \rightarrow 0} \frac{\text{covol}(Y + tX) - \text{covol}(Y)}{t}, \quad X, Y \in C^\infty(\Gamma). \quad (13)$$

LEMMA 3.4. *The function covol is C^∞ on $C^\infty(\Gamma)$, and for $h \in C_+^\infty(\Gamma)$, $X, Y \in C^\infty(\Gamma)$, we have:*

$$D_h \text{covol}(X) = - (X, \kappa^{-1}(h)), \quad (14)$$

$$D_h^2 \text{covol}(X, Y) = - (X, D_h \kappa^{-1}(Y)). \quad (15)$$

Moreover (14) is equivalent to

$$(X, D_h \kappa^{-1}(Y)) = (Y, D_h \kappa^{-1}(X)). \quad (16)$$

Proof. The second order differential operator κ^{-1} is smooth as the determinant is smooth [Ham82, 3.6.6]. Differentiating (12) we get

$$D_h \text{covol}(X) = -\frac{1}{d+1} ((X, \kappa^{-1}(h)) + (h, D_h \kappa^{-1}(X))), \quad (17)$$

but the bilinear form (\cdot, \cdot) is continuous for the seminorms $\|\cdot\|_n$ (recall that it suffices to check continuity in each variable [Rud91, 2.17]). It follows that covol is C^1 , and by iteration that it is C^∞ .

If (14) is true we get (15), and this expression is symmetric as covol is C^2 , so (16) holds.

Let us suppose that (16) is true. From (11), κ^{-1} is homogeneous of degree d , that gives $D_h \kappa^{-1}(h) = d\kappa^{-1}(h)$. Using this in (16) with $Y = h$ gives

$$d(X, \kappa^{-1}(h)) = (h, D_h \kappa^{-1}(X)).$$

Inserting this equation in (17) leads to (14).

A proof of (16) is done in [CY76] (for the case of C^2 functions on the sphere). See also [OS83] and reference therein for more generality. We will prove (14) following [Hör07]. From the definition of κ^{-1} , the map $D_h\kappa^{-1}(\cdot)$ is linear, hence from (17) $D_h\text{covol}(\cdot)$ is also linear, so by Lemma 3.3 it suffices to prove (14) for $X = h' \in C_+^\infty(\Gamma)$. We denote by K (resp. K') the Γ -convex body supported by h (resp. h') and by N (resp. N') its Gauss map. We have, for $\eta \in \mathcal{F}, \epsilon > 0$,

$$h(\eta) + \epsilon h'(\eta) = \langle \eta, N^{-1}(\eta) + \epsilon(N')^{-1}(\eta) \rangle_-$$

i.e. $h + \epsilon h'$ supports the hypersurface with position vector $N^{-1}(\eta) + \epsilon(N')^{-1}(\eta)$.

For a compact $U \subset \mathbb{R}^d$, if $f : U \rightarrow \mathbb{R}^{d+1}$ is a local parametrization of ∂K , let us introduce

$$F : U \times [0, \epsilon] \rightarrow \mathbb{R}^{d+1}, (y, t) \mapsto f(y) + t(N')^{-1}(N(f(y))).$$

It is a local parametrization of the set between the boundary of K and the boundary of $K + \epsilon K'$. Locally, its covolume (which corresponds to $\text{covol}(h + \epsilon h') - \text{covol}(h)$) is computed as

$$\int_{F(U \times [0, \epsilon])} d \text{ vol} = \int_0^\epsilon \int_U |\text{Jac} F| \, dy \, dt. \tag{18}$$

The Jacobian of F is equal to $\left((N')^{-1}(N(f(y))), \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_d} \right) + tR$ where R is a remaining term, and its determinant is equal to the determinant of $\left((N')^{-1}(N(f(y))), \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_d} \right)$ plus t times remaining terms. As $\left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_d} \right)$ form a basis of the tangent hyperplane of ∂K , and as N is normal to this hyperplane, the determinant is equal to $\langle (N')^{-1}(N(f(y))), N(f(y)) \rangle_- = h'(N(f(y)))$ times $|\text{Jac} f|$, plus t times remaining terms. The limit of (18) divided by ϵ when $\epsilon \rightarrow 0$ gives

$$\int_U h'(N(f(y))) |\text{Jac} f| \, dy = \int_{f(U)} h'(N) \, d \partial K.$$

The result follows by decomposing the boundary of K with suitable coordinate patches. □

The main result of this section is the following.

Theorem 3.5. *The second derivative of $\text{covol} : C^\infty(\Gamma) \rightarrow \mathbb{R}$ is positive definite. In particular the covolume of C_+^∞ Γ -convex bodies is strictly convex.*

Let us have a look at the case $d = 1$. In this case $\kappa^{-1} = r$, the unique radius of curvature. We parametrize the branch of the hyperbola by $(\sinh t, \cosh t)$, and h becomes a function from \mathbb{R} to \mathbb{R}_- . Then (8) reads

$$\kappa^{-1}(h)(t) = -h(t) + h''(t),$$

and, as h is Γ -invariant, we can consider κ^{-1} as a linear operator on the set of C^∞ functions on $[0, \ell]$, if ℓ is the length of the circle \mathbb{H}^1/Γ . Using integration by parts and the fact that h is ℓ -periodic, we get

$$D_h^2 \text{covol}(h, h) = - (h, \kappa^{-1}(h)) = - \int_0^\ell h \kappa^{-1}(h) = \int_0^\ell (h^2 + h'^2).$$

We will prove a more general version of Theorem 3.5 in the next section, using the theory of mixed-volume. The proof is based on the following particular case.

LEMMA 3.6. *Let h_0 be the support function of B^d (i.e. $h_0(\eta) = -1$). Then $D_{h_0}^2 \text{covol}$ is positive definite.*

Proof. Let $X \in C^\infty(\Gamma)$. From the definition (11) of κ^{-1}

$$D_h \kappa^{-1}(X) = \kappa^{-1}(h) \sum_{i=1}^d r_i^{-1}(h) D_h r_i(X)$$

and as $r_i(h_0) = 1$,

$$D_{h_0} \kappa^{-1}(X) = \sum_{i=1}^d D_{h_0} r_i(X).$$

Differentiating (8) on both side at h_0 and passing to the quotient, the equation above gives

$$D_{h_0} \kappa^{-1}(X) = -dX + \Delta_M X,$$

where Δ_M is the Laplacian on $M = \mathbb{H}^d/\Gamma$. From (15),

$$D_{h_0}^2 \text{covol}(X, X) = d(X, X) - (\Delta_M X, X),$$

which is positive by property of the Laplacian, as $M = \mathbb{H}^d/\Gamma$ is compact. \square

3.3 Smooth Minkowski theorem. One can ask if, given a positive function f on a hyperbolic compact manifold $M = \mathbb{H}^d/\Gamma$, it is the Gauss curvature of a C_+^2 convex Fuchsian surface and if the former one is unique. By Lemma 3.2 and definition of the Gauss curvature, the question reduces to know if there exists a (unique) function h on M such that, in an orthogonal frame on M ,

$$f = \det((\nabla^2 h)_{ij} - h \delta_{ij})$$

and

$$((\nabla^2 h)_{ij} - h \delta_{ij}) > 0.$$

This PDE problem is solved in [OS83] in the smooth case. Their main result (Theorem 3.4) can be written as follows.

Theorem 3.7. *Let Γ be a Fuchsian group, $f : \mathbb{H}^d \rightarrow \mathbb{R}_+$ be a positive C^∞ Γ -invariant function.*

There exists a unique C_+^∞ Γ -convex body with Gauss curvature f .

3.4 Mixed curvature and mixed-covolume The determinant is a homogeneous polynomial of degree d , and we denote by $\det(\cdot, \dots, \cdot)$ its polar form, that is the unique symmetric d -linear form such that

$$\det(A, \dots, A) = \det(A)$$

for any $d \times d$ symmetric matrix A (see for example Appendix A in [Hör07]). We will need the following key result.

Theorem 3.8 ([Ale96, p. 125]). *Let A, A_3, \dots, A_d be positive definite $d \times d$ matrices and Z be a symmetric matrix. Then*

$$\det(Z, A, A_3, \dots, A_d) = 0 \Rightarrow \det(Z, Z, A_3, \dots, A_d) \leq 0,$$

and equality holds if and only if Z is identically zero.

For any orthonormal frame on $M = \mathbb{H}^d/\Gamma$ and for $X_k \in C^\infty(\Gamma)$, let us denote

$$X_k'' := (\nabla^2 X_k)_{ij} - X_k \delta_{ij}$$

and let us introduce the *mixed curvature*

$$\kappa^{-1}(X_1, \dots, X_d) := \det(X_1'', \dots, X_d'').$$

As $\text{covol}(X) = -\frac{1}{d+1} (X, \kappa^{-1}(X))$, covol is a homogeneous polynomial of degree $d+1$. Its polar form $\text{covol}(\cdot, \dots, \cdot)$ ($(d+1)$ entries) is the *mixed-covolume*.

LEMMA 3.9. *We have the following equalities, for $X_i \in C^\infty(\Gamma)$.*

- (i) $D_{X_2}^{d-1} \kappa^{-1}(X_3, \dots, X_{d+1}) = d! \kappa^{-1}(X_2, \dots, X_{d+1})$,
- (ii) $D_{X_1} \text{covol}(X_2) = (d+1) \text{covol}(X_2, X_1, \dots, X_1)$,
- (iii) $D_{X_1}^2 \text{covol}(X_2, X_3) = (d+1) d \text{covol}(X_2, X_3, X_1, \dots, X_1)$,
- (iv) $D_{X_1}^d \text{covol}(X_2, \dots, X_{d+1}) = (d+1)! \text{covol}(X_1, \dots, X_{d+1})$,
- (v) $\text{covol}(X_1, \dots, X_{d+1}) = -\frac{1}{d+1} (X_1, \kappa^{-1}(X_2, \dots, X_{d+1}))$.

Proof. (i) and (iv) are proved by induction on the order of the derivative, using the definition of directional derivative and the expansion of the multilinear forms. (ii) and (iii) are obtained by the way. (v) follows from (14), (i) and (iv). \square

COROLLARY 3.10. *For $h_i \in C_+^\infty(\Gamma)$, $\text{covol}(h_1, \dots, h_{d+1})$ is positive.*

Proof. As $h_i \in C_+^\infty(\Gamma)$, h_i'' is positive definite, hence $\kappa^{-1}(h_2, \dots, h_{d+1}) > 0$ [Ale96, (5) p. 122]. The result follows from (v) because $h_1 < 0$. \square

Due to (iii) of the preceding lemma, the following result implies Theorem 3.5.

Theorem 3.11. *For any h_1, \dots, h_{d-1} in $C_+^\infty(\Gamma)$, the symmetric bilinear form on $(C^\infty(\Gamma))^2$*

$$\text{covol}(\cdot, \cdot, h_1, \dots, h_{d-1})$$

is positive definite.

Proof. We use a continuity method. We consider the paths $h_i(t) = th_i + (1-t)h_0$, $i = 1, \dots, d-1$, $t \in [0, 1]$, where h_0 is the (quotient of the) support function of B^d and we denote

$$\text{covol}_t(\cdot, \cdot) := \text{covol}(\cdot, \cdot, h_1(t), \dots, h_{d-1}(t)).$$

The result follows from the facts:

- (i) covol_0 is positive definite,
- (ii) if, for each $t_0 \in [0, 1]$, covol_{t_0} is positive definite, then covol_t is positive definite for t near t_0 ,
- (iii) if $t_n \in [0, 1]$ with $t_n \rightarrow t_0$ and covol_{t_n} is positive definite, then covol_{t_0} is positive definite.

(i) is Lemma 3.6. Let t_0 as in (ii). By Lemma 3.12, each $\kappa^{-1}(\cdot, h_1(t), \dots, h_{d-1}(t))$ inherits standard properties of elliptic self-adjoint operators on compact manifolds (see for example [Nic07]), and we can apply [Kat95, Theorem 3.9 p. 392]: as the deformation of the operators is polynomial in t , the eigenvalues change analytically with t , for t near t_0 . In particular if t is sufficiently close to t_0 , the eigenvalues remain positive and (ii) holds.

Let t_n be as in (iii). For any non zero $X \in C^\infty(\Gamma)$ we have $\text{covol}_{t_n}(X, X) > 0$ with

$$\text{covol}_{t_n}(X, X) = \int_M X \kappa^{-1}(X, (1-t_n)h_0 + t_n h_1, \dots, (1-t_n)h_0 + t_n h_{d-1}) dM.$$

As κ^{-1} is multilinear and as $t_n < 1$, it is easy to see that the function in the integrand above is bounded by a function (of the kind $X \sum |\kappa^{-1}(X, *, \dots, *)|$ where each $*$ is h_0 or a h_i) which does not depend on n and is continuous on the compact M . By Lebesgue's dominated convergence theorem, $\text{covol}_{t_0}(X, X) \geq 0$, and by Lemma 3.13 $\text{covol}_{t_0}(X, X) > 0$, and (iii) is proved. \square

LEMMA 3.12. *For any h_1, \dots, h_{d-1} in $C_+^\infty(\Gamma)$, the operator $\kappa^{-1}(\cdot, h_1, \dots, h_{d-1})$ is formally self-adjoint linear second order elliptic.*

Proof. It is formally self-adjoint because of the symmetry of the mixed-covolume. It is clearly second order linear. Let $Z \in C^\infty(\Gamma)$. From properties of the mixed determinant [Ale96, p. 121], $\kappa^{-1}(Z, h_1, \dots, h_{d-1})$ can be written, for an orthonormal frame on M ,

$$\sum_{i,j=1}^d \det(h_1'', \dots, h_{d-1}'')_{ij} ((\nabla^2 Z)_{ij} - Z \delta_{ij})$$

where $\det(h_1'', \dots, h_{d-1}'')_{ij}$ is, up to a constant factor, the mixed determinant of the matrices obtained from the h_k'' by deleting the i th row and the j th column. Let us consider local coordinates on M around a point p such that at p , $\kappa^{-1}(Z, h_1, \dots, h_{d-1})$

has the expression above. By definition of $C_+^\infty(\Gamma)$, h_k'' are positive definite at p and then at p

$$\sum_{i,j=1}^d \det(h_1'', \dots, h_{d-1}'')_{ij} x_i x_j$$

is positive definite [Ale96, Lemma II p. 124]. □

LEMMA 3.13. *For any h_1, \dots, h_{d-1} in $C_+^\infty(\Gamma)$, the symmetric bilinear form*

$$\text{covol}(\cdot, \cdot, h_1, \dots, h_{d-1})$$

has trivial kernel.

Proof. Suppose that Z belongs to the kernel of $\text{covol}(\cdot, \cdot, h_1, \dots, h_{d-1})$. As (\cdot, \cdot) is an inner product, Z belongs to the kernel of $\kappa^{-1}(\cdot, h_1, \dots, h_{d-1})$:

$$\det(Z'', h_1'', \dots, h_{d-1}'') = 0.$$

As h_i'' are positive definite matrices, by definition of $C_+^\infty(\Gamma)$, Theorem 3.8 implies that

$$\det(Z'', Z'', h_2'', \dots, h_{d-1}'') \leq 0$$

so

$$0 = \text{covol}(Z, Z, h_1, \dots, h_{d-1}) = - \int_M h_1 \kappa^{-1}(Z, Z, h_2, \dots, h_{d-1}) \leq 0$$

but $h_1 < 0$ hence

$$\det(Z'', Z'', h_2'', \dots, h_{d-1}'') = 0,$$

and Theorem 3.8 says that $Z'' = 0$. Consider the 1-homogeneous extension \tilde{Z} of the Γ invariant map on \mathbb{H}^d defined by Z . From Section 3.1 it follows that the Hessian of \tilde{Z} in \mathcal{F} is zero, hence that \tilde{Z} is affine. By invariance \tilde{Z} must be constant, and by homogeneity $\tilde{Z} = 0$ hence $Z = 0$. □

Remark on Fuchsian Hedgehogs. If we apply Cauchy–Schwarz inequality to the inner product of Theorem 3.11, we get a “reversed Alexandrov–Fenchel inequality” (see Theorem 5.5) for C_+^∞ convex bodies, but also for any smooth function h on the hyperbolic manifold \mathbb{H}^d/Γ . From Lemma 3.3 there exist two elements h_1, h_2 of $C_+^\infty(\Gamma)$ with $h = h_1 - h_2$. Hence h can be seen as the “support function” of the (maybe non convex) hypersurface made of the points $\nabla_\eta(H_1 - H_2), \eta \in \mathcal{F}$. For example if h_1 and h_2 are the support functions of respectively B_{t_1} and B_{t_2} , then h is the support function of a pseudo-sphere in \mathcal{F} if $t_1 - t_2 > 0$, of a point (the origin) if $t_1 - t_2 = 0$ and of a pseudo-sphere in the past cone if $t_1 - t_2 < 0$.

More generally, we could introduce “Fuchsian hedgehogs”, whose “support functions” are difference of support functions of two Γ -convex bodies. They form the vector space in which the support functions of Γ -convex bodies naturally live. In the Euclidean space, they were introduced in [LLR88]. An Euclidean analog of the reversed Alexandrov–Fenchel inequality for smooth Fuchsian hedgehogs described above is done in [Mar99], among other results. It would be interesting to know if other results about hedgehogs have a Fuchsian analogue.

4 Polyhedral Case

The classical analogue of this section comes from [Ale37] (see [Ale96]). See also [Sch93] and [Ale05]. The toy example $d = 1$ is considered in [Fil11b].

4.1 Support vectors

Definition of Fuchsian convex polyhedron. The notation a^\perp will represent the affine hyperplane over the vector hyperplane orthogonal to the vector a and passing through a :

$$a^\perp = \{x \in \mathbb{R}^{d+1} \mid \langle x, a \rangle_- = \langle a, a \rangle_-\}. \quad (19)$$

DEFINITION 4.1. Let $R = (\eta_1, \dots, \eta_n)$, $n \geq 1$, with η_i (pairwise non-collinear) vectors in the future cone \mathcal{F} , and let Γ be a Fuchsian group. A Γ -convex polyhedron is the boundary of the intersection of the half-spaces bounded by the hyperplanes

$$(\gamma\eta_i)^\perp, \quad \forall \gamma \in \Gamma, \forall i = 1, \dots, n,$$

such that the vectors η_i are inward pointing.

See Figure 2 for a simple example.

LEMMA 4.2. A Γ -convex polyhedron P

- (i) is a Γ -convex body,
- (ii) has a countable number of facets,
- (iii) is locally finite,
- (iv) each face is a convex Euclidean polytope.

Here convex polytope means convex compact polyhedron.

Proof. We denote by P_i the Γ -convex polyhedron made from the vector η_i and the group Γ . We will prove the lemma for P_i . The general case follows because P is the intersection of a finite number of P_i . All the elements of $\Gamma\eta_i$ belong to $\mathbb{H}_{t_i}^d$, on which Γ acts cocompactly. Up to a homothety, it is more suitable to consider that $\mathbb{H}_{t_i}^d$ is $\mathbb{H}_1^d = \mathbb{H}^d$.

Let $a \in \Gamma\eta_i$ and $D_a(\Gamma)$ be the Dirichlet region (see (1)). Recall that $D_a(\Gamma)$ are convex compact polyhedra in \mathbb{H}^d , and that the set of the Dirichlet regions D_a , for

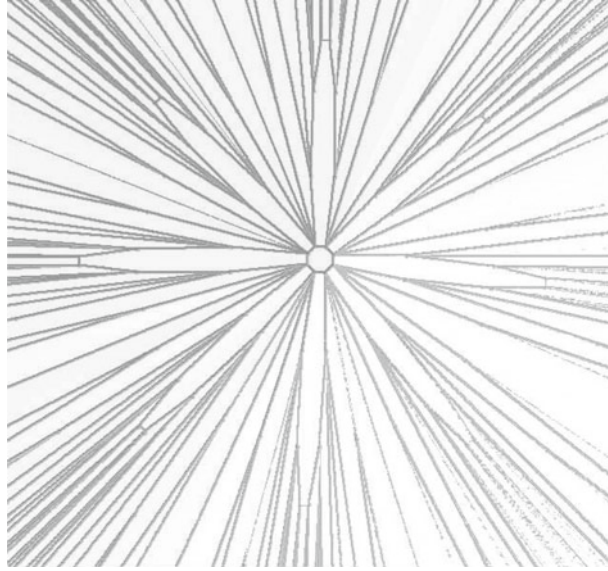


Figure 2: A piece of a Γ -convex polyhedron in $d = 2$ seen from the bottom. It is made with the orbit of $(0, 0, 1)$ for the Fuchsian group having a regular octagon as fundamental domain in \mathbb{H}^2

all $a \in \Gamma\eta_i$, is a locally finite tessellation of \mathbb{H}^d . Using (20), the Dirichlet region can be written

$$D_a(\Gamma) = \{p \in \mathbb{H}^d \mid \langle a, p \rangle_- \geq \langle \gamma a, p \rangle_-, \forall \gamma \in \Gamma \setminus \{Id\}\}.$$

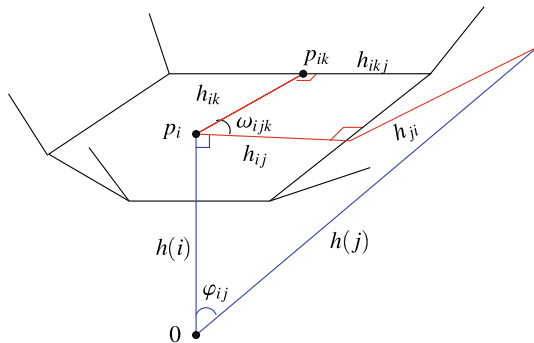
Let $a_1, a_2 \in \Gamma\eta_i$ such that $D_{a_1}(\Gamma)$ and $D_{a_2}(\Gamma)$ have a common facet. This facet is contained in the intersection of \mathbb{H}^d with the hyperplane

$$\{p \in \mathbb{R}^{d+1} \mid \langle a_1, p \rangle_- = \langle a_2, p \rangle_-\},$$

and this hyperplane also contains $a_1^\perp \cap a_2^\perp$ by (19). It follows that vertices of P_i (codimension $(d + 1)$ faces) project along rays from the origin onto the vertices of the Dirichlet tessellation. In particular the vertices are in \mathcal{F} , so $P_i \subset \mathcal{F}$, because it is the convex hull of its vertices [Sch93, 1.4.3] and \mathcal{F} is convex. In particular P_i is a Γ -convex body due to Definition 2.1. And codimension k faces of P_i projects onto codimension k faces of the Dirichlet tessellation, so P_i is locally finite with a countable number of facets.

Facets of P_i are closed, as they project onto compact sets. In particular they are bounded as contained in \mathcal{F} hence compact. They are convex polytopes by construction, and Euclidean as contained in space-like planes. Higher codimension faces are convex Euclidean polytopes as intersections of convex Euclidean polytopes. \square

Support numbers. The extended support function of a Γ -convex polyhedron P is piecewise linear (it is linear on each solid angle determined by the normals of the

Figure 3: Supports numbers of a Γ -convex polyhedron

support planes at a vertex), it is why the data of the extended support function on each inward unit normal of the facets suffices to determine it. If η_i is such a vector and h is the support function of P , we call the positive number

$$h(i) := -h(\eta_i)$$

the i th support number of P .

The facet with normal η_i is denoted by F_i . Two adjacent facets F_i and F_j meet at a codimension 2 face F_{ij} . If three facets F_i, F_j, F_k meet at a codimension 3 face, then this face is denoted by F_{ijk} . We denote by φ_{ij} the hyperbolic distance between η_i and η_j , given by (see for example [Rat06, (3.2.2)])

$$-\cosh \varphi_{ij} = \langle \eta_i, \eta_j \rangle_- . \quad (20)$$

Let p_i be the foot of the perpendicular from the origin to the hyperplane \mathcal{H}_i containing the facet F_i . In \mathcal{H}_i , let p_{ij} be the foot of the perpendicular from p_i to F_{ij} . We denote by h_{ij} the signed distance from p_i to p_{ij} : it is non negative if p_i is in the same side of F_j than P . See Figure 3.

For each i , h_{ij} are the support numbers of the convex Euclidean polytope F_i . (\mathcal{H}_i is identified with the Euclidean space \mathbb{R}^d , with p_i as the origin.) If we denote by ω_{ijk} the angle between $p_i p_{ij}$ and $p_i p_{ik}$, it is well-known that [Sch93, (5.1.3)]

$$h_{ikj} = \frac{h_{ij} - h_{ik} \cos \omega_{ijk}}{\sin \omega_{ijk}} . \quad (21)$$

We have a similar formula in Minkowski space [Fil11b]:

$$h_{ij} = -\frac{h(j) - h(i) \cosh \varphi_{ij}}{\sinh \varphi_{ij}} . \quad (22)$$

In particular,

$$\frac{\partial h_{ij}}{\partial h(j)} = -\frac{1}{\sinh \varphi_{ij}} , \quad (23)$$

$$\frac{\partial h_{ij}}{\partial h(i)} = \frac{\cosh \varphi_{ij}}{\sinh \varphi_{ij}} . \quad (24)$$

If $h(i) = h(j)$ and if the quadrilateral is deformed under this condition, then

$$\frac{\partial h_{ij}}{\partial h(i)} = \frac{\cosh \varphi_{ij} - 1}{\sinh \varphi_{ij}}. \tag{25}$$

Space of polyhedra with parallel facets. Let P be a Γ -convex polyhedron. We label the facets of P in a fundamental domain for the action of Γ . This set of label is denoted by \mathcal{I} , and $\Gamma\mathcal{I}$ labels all the facets of P . Let $R = (\eta_1, \dots, \eta_n)$ be the inward unit normals of the facets of P labeled by \mathcal{I} .

We denote by $\mathcal{P}(\Gamma, R)$ the set of Γ -convex polyhedra with inward unit normals belonging to the set R . By identifying a Γ -convex polyhedron with its support numbers labeled by \mathcal{I} , $\mathcal{P}(\Gamma, R)$ is a subset of \mathbb{R}^n . (The corresponding vector of \mathbb{R}^n is the *support vector* of the polyhedron.) Note that this identification does not commute with the sum. Because the sum of two piecewise linear functions is a piecewise linear function, the Minkowski sum of two Γ -convex polyhedra is a Γ -convex polyhedron. (More precisely, the linear functions under consideration are of the form $\langle \cdot, v \rangle_-$, with v a vertex of a polyhedron, hence a future time-like vector, and the sum of two future time-like vectors is a future time-like vector.) But even if the two polyhedra have parallel facets, new facets can appear in the sum. Later we will introduce a class of polyhedra such that the support vector of the Minkowski sum is the sum of the support vectors.

LEMMA 4.3. *The set $\mathcal{P}(\Gamma, R)$ is a non-empty open convex cone of \mathbb{R}^n .*

Proof. The condition that the hyperplane supported by η_j contains a facet of the polyhedron with support vector h can be written as

$$\exists x \in \mathbb{R}^{d+1}, \forall i \in \Gamma\mathcal{I}, i \neq j, \langle \eta_i, x \rangle_- < -h(i) \text{ and } \langle \eta_j, x \rangle_- = -h(j).$$

By (20) $\mathcal{P}(\Gamma, R)$ always contains the vector $(1, \dots, 1)$. The set is clearly open as a facet can't disappear for any sufficiently small deformation. It is also clearly invariant under homotheties of positive scale factor. So to prove that $\mathcal{P}(\Gamma, R)$ is a convex cone it suffices to check that if h and h' belongs to $\mathcal{P}(\Gamma, R)$ then $h + h'$ belongs to $\mathcal{P}(\Gamma, R)$. It is immediate from the above characterization. \square

4.2 Covolume of convex Fuchsian polyhedra. Let F be a facet of a Γ -convex polyhedron P , contained in a space-like hyperplane \mathcal{H} , with support number h . For the induced metric, \mathcal{H} is isometric to the Euclidean space \mathbb{R}^d , in which F is a convex polytope, with volume $A(F)$. We call $A(F)$ the *area* of the facet. Let C be the cone in \mathbb{R}^{d+1} over P with apex the origin. Its volume $V(C)$ is invariant under the action of an orientation and time-orientation preserving linear isometry (they have determinant 1), hence to compute $V(C)$ we can suppose that \mathcal{H} is an horizontal hyperplane (constant last coordinate). For horizontal hyperplanes, the induced metric is the same if the ambient space is endowed with the standard Lorentzian metric or with the standard Euclidean metric. So the well-known formula applies:

$$V(C) = \frac{1}{d+1} hA(F),$$

and then

$$\text{covol}(P) = \frac{1}{d+1} \sum_{i \in \mathcal{I}} h(i) A(F_i).$$

Identifying P with its support vector h , if $\langle \cdot, \cdot \rangle$ is the usual inner product of \mathbb{R}^n , we have

$$\text{covol}(h) = \frac{1}{d+1} \langle h, A(h) \rangle \quad (26)$$

where $A(h)$ is the vector formed by the area of the facets $A(F_i)$.

LEMMA 4.4. *The function covol is C^2 on \mathbb{R}^n , and for $h \in \mathcal{P}(\Gamma, R)$, $X, Y \in \mathbb{R}^n$, we have:*

$$D_h \text{covol}(X) = \langle X, A(h) \rangle, \quad (27)$$

$$D_h^2 \text{covol}(X, Y) = \langle X, D_h A(Y) \rangle. \quad (28)$$

Moreover (27) is equivalent to

$$\langle X, D_h A(Y) \rangle = \langle Y, D_h A(X) \rangle. \quad (29)$$

Proof. Let P be the polyhedron with support function $h \in \mathcal{P}(\Gamma, R)$. Let F_i be a facet of P , with support numbers h_{i1}, \dots, h_{im} . If V_E is the d Euclidean volume, it is well-known that [Ale05, 8.2.3]

$$\frac{\partial V_E(F_i)}{\partial h_{ik}} = L_{ik} \quad (30)$$

where L_{ik} is the area of the facet of F_i with support number h_{ik} (for $d = 1$, one has 1 instead of L_{ik}). $A(F_i)$ is not exactly as $V_E(F_i)$, because it is a function of h , and, when varying a $h(j)$, a new facet of F_i can appear, as well as a new support number h_{ij} of F_i . Actually many new facets can appear, as many as hyperplanes with normals $\Gamma \eta_j$ meeting F_i . One has to consider F_i as also supported by h_{ij} (and eventually some orbits). In this case, $L_{ij} = 0$, and the variation of the volume is still given by formula (30). So even if the combinatorics of P changes under small change of a support number, there is no contribution to the change of the volume of the facets. So (30) gives

$$\frac{\partial A(F_i)}{\partial h_{ik}} = L_{ik}. \quad (31)$$

We denote by $E_i^j \subset \Gamma \mathcal{I}$ is the set of indices $k \in \Gamma j$ such that F_k is adjacent to F_i along a codimension 2 face. It can be empty. But for example if \mathcal{I} is reduced to a single element i , E_i^i is the set of facets adjacent to F_i along a codimension 2 face. If $j \in \mathcal{I} \setminus \{i\}$ we get

$$\frac{\partial A(F_i)}{\partial h(j)} = \sum_{k \in E_i^j} \frac{\partial A(F_i)}{\partial h_{ik}} \frac{\partial h_{ik}}{\partial h(j)}.$$

From (23) and (31) it follows that

$$\frac{\partial A(F_i)}{\partial h(j)} = - \sum_{k \in E_i^j} \frac{L_{ik}}{\sinh \varphi_{ik}}. \tag{32}$$

For the diagonal terms:

$$\begin{aligned} \frac{\partial A(F_i)}{\partial h(i)} &= \sum_{j \in \mathcal{I} \setminus \{i\}} \sum_{k \in E_i^j} \frac{\partial A(F_i)}{\partial h_{ik}} \frac{\partial h_{ik}}{\partial h(i)} + \sum_{k \in E_i^i} \frac{\partial A(F_i)}{\partial h_{ik}} \frac{\partial h_{ik}}{\partial h(i)} \\ &\stackrel{(31),(24),(25)}{=} \sum_{j \in \mathcal{I} \setminus \{i\}} \sum_{k \in E_i^j} \cosh \varphi_{ik} \frac{L_{ik}}{\sinh \varphi_{ik}} + \sum_{k \in E_i^i} L_{ik} \frac{\cosh \varphi_{ik} - 1}{\sinh \varphi_{ik}}. \end{aligned} \tag{33}$$

These expressions are continuous with respect to h , even if the combinatorics changes. So A is C^1 and from (26) covol is C^2 .

If (27) is true, we get (28), and this expression is symmetric as covol is C^2 , so (29) holds. Let us suppose that (29) is true. As made of volumes of convex polytopes of \mathbb{R}^d , A is homogeneous of degree d so by Euler homogeneous theorem $D_h A(h) = dA(h)$. Using this in (29) with $Y = h$ gives $d \langle X, A(h) \rangle = \langle h, D_h A(X) \rangle$. Now differentiating (26) gives $D_h \text{covol}(X) = \frac{1}{d+1} \langle X, A(h) \rangle + \frac{1}{d+1} \langle h, D_h A(X) \rangle$. Inserting the preceding equation leads to (27).

Let us prove (29). If e_1, \dots, e_n is the standard basis of \mathbb{R}^n , it suffices to prove (29) for $X = e_i$ and $Y = e_j$, $i \neq j$ i.e that the gradient of A is symmetric. The sum in (32) means that, in $\partial P/\Gamma$, each times the i th polytope meets the j th polytope along a codimension 2 face, we add the quantity $\frac{L_{ik}}{\sinh \varphi_{ik}}$, which is symmetric in its arguments. Hence the gradient of A is symmetric. \square

Let us consider the simplest case of Γ -convex polyhedra in the Minkowski plane, with only one support number $h \in \mathbb{R}$. Then by (22) $\text{covol}(h)$ is equal to h^2 times a positive number, in particular it is a strictly convex function. This is always true.

Theorem 4.5. *The Hessian of $\text{covol} : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite.*

Recall that we are looking at the covolume on a space of support vectors, and not on a space of polyhedra (the sum is not the same).

Proof. Due to (28) it suffices to study the Jacobian of A . The elements off the diagonal are non-positive due to (32). Note that the formula is also correct if E_i^j is empty. The diagonal terms (33) are positive, as any facet F_i has an adjacent facet. As $\cosh x > 1$ for $x \neq 0$, (33) and (32) lead to

$$\frac{\partial A(F_i)}{\partial h(i)} > \sum_{j \in \mathcal{I} \setminus \{i\}} \left| \frac{\partial A(F_i)}{\partial h(j)} \right| > 0$$

that means that the Jacobian is strictly diagonally dominant with positive diagonal entries, hence positive definite, see for example [Var00, 1.22]. \square

4.3 Polyhedral Minkowski theorem. We use a classical continuity method, although its Euclidean analog is more often proved using a variational method.

Theorem 4.6 (Minkowski Theorem). *Let Γ be a Fuchsian group, $R = (\eta_1, \dots, \eta_n)$ be a set of pairwise non collinear unit future time-like vectors of the Minkowski space contained in a fundamental domain of Γ , and let (f_1, \dots, f_n) be positive real numbers.*

There exists a unique Γ -convex polyhedron with inward unit normals η_i such that the facet orthogonal to η_i has area f_i .

Theorem 4.6 is equivalent to say that the map Φ from $\mathcal{P}(\Gamma, R)$ to $(\mathbb{R}_+)^n$ which associates to each $(h_1, \dots, h_n) \in \mathcal{P}(\Gamma, R)$ the facet areas $(A(F_1), \dots, A(F_n))$ is a bijection. By Lemma 4.4, Theorem 4.5 and local inverse theorem, Φ is locally invertible. So Φ is a local homeomorphism by the invariance of domain theorem. Lemma 4.7 below says that Φ is proper. As $(\mathbb{R}_+)^n$ is connected, it follows that Φ is surjective, hence a covering map. But the target space $(\mathbb{R}_+)^n$ is simply connected and $\mathcal{P}(\Gamma, R)$ is connected (Lemma 4.3), so Φ is a homeomorphism, in particular bijective, and Theorem 4.6 is proved.

LEMMA 4.7. *The map Φ is proper: Let $(a_\alpha)_{\alpha \in \mathbb{N}}$ be a converging sequence of $(\mathbb{R}_+)^n$ such that for all α , there exists $h_\alpha = (h_\alpha(1), \dots, h_\alpha(n)) \in \mathcal{P}(\Gamma, R)$ with $\Phi(h_\alpha) = a_\alpha$. Then a subsequence of $(h_\alpha)_\alpha$ converges in $\mathcal{P}(\Gamma, R)$.*

Proof. Let $\alpha \in \mathbb{N}$ and suppose that $h_\alpha(i)$ is the largest component of h_α . For any support number $h_\alpha(j)$, $j \in \Gamma\mathcal{I}$, of a facet adjacent to the one supported by $h_\alpha(i)$, as $h_\alpha(i) \geq h_\alpha(j)$, (22) gives:

$$h_{ij}^\alpha = \frac{h_\alpha(i) \cosh \varphi_{ij} - h_\alpha(j)}{\sinh \varphi_{ij}} \geq h_\alpha(i) \frac{\cosh \varphi_{ij} - 1}{\sinh \varphi_{ij}}.$$

As Γ acts cocompactly on \mathbb{H}^d , for any $j \in \Gamma\mathcal{I}$, φ_{ij} is bounded from below by a positive constant. Moreover the function $x \mapsto \frac{\cosh x - 1}{\sinh x}$ is increasing, then there exists a positive number λ_i , depending only on i , such that

$$h_{ij}^\alpha \geq h_\alpha(i) \lambda_i.$$

As the sequence of areas of the facets is supposed to converge, there exists positive numbers A_i^+ and A_i^- such that $A_i^+ \geq A(F_i^\alpha) \geq A_i^-$, where $A(F_i^\alpha)$ is the area of the facet F_i^α supported by $h_\alpha(i)$. If Per_i^α (resp. Per_i) is the Euclidean $(d-1)$ volume of the hypersphere bounding the ball with Euclidean d volume $A(F_i^\alpha)$ (resp. A_i^-), the isoperimetric inequality gives [BZ88, 10.1]

$$\sum_j L_{ij}^\alpha \geq \text{Per}_i^\alpha \geq \text{Per}_i,$$

where the sum is on the facets adjacent to F_i^α and L_{ij}^α is the $(d - 1)$ volume of the codimension 2 face between F_i^α and F_j^α . We get

$$A_i^+ \geq A(F_i^\alpha) = \frac{1}{d} \sum_j h_{ij}^\alpha L_{ij}^\alpha \geq h_\alpha(i) \lambda_i \frac{1}{d} \sum_j L_{ij}^\alpha \geq h_\alpha(i) \frac{\lambda_i \text{Per}_i}{d}.$$

As $h_\alpha(i)$ is the largest component of h_α , all the support numbers are bounded from above by a constant which does not depend on α . Moreover each component of h_α is positive, hence all the components of the elements of the sequence $(h_\alpha)_\alpha$ are bounded from above and below, so there exists a subsequence $(h_{\phi(\alpha)})_{\phi(\alpha)}$ converging to $(h(1), \dots, h(n))$, where $h(i)$ is a non-negative number.

Suppose that the limit of $(h_{\phi(\alpha)}(i))_{\phi(\alpha)}$ is zero. Let $h_{\phi(\alpha)}(j)$ be the support number of a facet adjacent to $F_i^{\phi(\alpha)}$. If $\phi(\alpha)$ is sufficiently large, $h_{\phi(\alpha)}(j)$ is arbitrary close to $h(j)$, which is a non-negative number, and $h_{\phi(\alpha)}(i)$ is arbitrary close to 0. By (22), h_{ij}^α is a non-positive number. So all the support numbers of $F_i^{\phi(\alpha)}$ are non-positive, hence the d volume of $F_i^{\phi(\alpha)}$ is non-positive, that is impossible. It follows easily that $(h_{\phi(\alpha)}(i))_{\phi(\alpha)}$ converges in $\mathcal{P}(\Gamma, R)$. □

4.4 Mixed face area and mixed-covolume. Let us recall some basic facts about convex polytopes in Euclidean space (with non empty interior). A convex polytope of \mathbb{R}^d is *simple* if each vertex is contained in exactly d facets. Each face of a simple convex polytope is a simple convex polytope. The *normal fan* of a convex polytope is the decomposition of \mathbb{R}^d by convex cones defined by the outward unit normals to the facets of the polytope (each cone corresponds to one vertex). Two convex polytopes are *strongly isomorphic* if they have the same normal fan. The Minkowski sum of two strongly isomorphic simple polytopes is a simple polytope strongly isomorphic to the previous ones. Moreover the support vector of the Minkowski sum is the sum of the support vectors.

Let Q be a simple convex polytope in \mathbb{R}^d with n facets. The set of convex polytopes of \mathbb{R}^d strongly isomorphic to Q is a convex open cone in \mathbb{R}^n . The Euclidean volume V_E is a polynomial of degree d on this set, and its polarization $V_E(\cdot, \dots, \cdot)$ is the *mixed-volume*. The coefficients of the volume depend on the combinatorics, it's why we have to restrict ourselves to simple strongly isomorphic polytopes. The following result is an equivalent formulation of the Alexandrov–Fenchel inequality.

Theorem 4.8 ([Ale96, Sch93]). *Let Q, Q_3, \dots, Q_d be strongly isomorphic simple convex polytopes of \mathbb{R}^d with n facets and $Z \in \mathbb{R}^n$. Then*

$$V_E(Z, Q, Q_3, \dots, Q_d) = 0 \Rightarrow V_E(Z, Z, Q_3, \dots, Q_d) \leq 0$$

and equality holds if and only if Z is the support vector of a point.

We identify a support hyperplane of an element of $\mathcal{P}(\Gamma, R)$ with the Euclidean space \mathbb{R}^d by performing a translation along the ray from the origin orthogonal to

the hyperplane. In this way we consider all facets of elements of $\mathcal{P}(\Gamma, R)$ lying in parallel hyperplanes as convex polytopes in the same Euclidean space \mathbb{R}^d .

The definition of strong isomorphy and simplicity extend to Γ -convex polyhedra, considering them as polyhedral hypersurface in the ambient vector space. Note that the simplest examples of Euclidean convex polytopes, the simplices, are simple, but the simplest examples of Γ -convex polyhedra, those defined by only one orbit, are not simple (if $d > 1$). Let us formalize the definition of strong isomorphy. The *normal cone* $N(P)$ of a convex Γ -polyhedron P is the decomposition of \mathcal{F} by convex cones defined by the inward normals to the facets of P . It is the minimal decomposition of \mathcal{F} such that the extended support functions of P is the restriction of a linear form on each part. If the normal fan $N(Q)$ subdivides $N(P)$, then we write $N(Q) > N(P)$. Note that

$$N(P + Q) > N(P).$$

Two convex Γ -polyhedron P and Q are *strongly isomorphic* if $N(P) = N(Q)$. If P is simple, we denote by $[P]$ the subset of $\mathcal{P}(\Gamma, R)$ made of polyhedra strongly isomorphic to P .

LEMMA 4.9. *All elements of $[P]$ are simple and $[P]$ is an open convex cone of \mathbb{R}^n .*

Proof. The fact that all elements of $[P]$ are simple and that $[P]$ is open are classical, see for example [Ale37]. The only difference with the Euclidean convex polytopes case is that, around a vertex, two facets can belong to the same orbit for the action of Γ , hence when one wants to slightly move a facet adjacent to a vertex, one actually moves two (or more) facets. But this does not break the simplicity, nor the strong isomorphy class. Moreover $[P]$ is a convex cone as the sum of two functions piecewise linear on the same decomposition of \mathcal{F} gives a piecewise linear function on the same decomposition. \square

Suppose that P is simple, has n facets (in a fundamental domain), and let $h_1, \dots, h_{d+1} \in [P]$ (support vectors of polyhedra strongly isomorphic to P). Let us denote by $F_k(i)$ the i th facet of the polyhedron with support vector h_k , and let $h(F_k(i))$ be its support vector ($F_k(i)$ is seen as a convex polytope in \mathbb{R}^d). The entries of $h(F_k(i))$ have the form (22) so the map $h_k \mapsto h(F_k(i))$ is linear. This map can be defined formally for all $Z \in \mathbb{R}^n$ using (22). The *mixed face area* $A(h_2, \dots, h_{d+1})$ is the vector formed by the entries $V_E(h(F_2(i)), \dots, h(F_{d+1}(i)))$, $i = 1, \dots, n$. Together with (26), this implies that covol is a $(d+1)$ -homogeneous polynomial, and we call *mixed-covolume* its polarization $\text{covol}(\cdot, \dots, \cdot)$. Note that covol is C^∞ on $[P]$.

LEMMA 4.10. *We have the following equalities, for $X_i \in \mathbb{R}^n$.*

- (i) $D_{X_2}^{d-1} A(X_3, \dots, X_{d+1}) = d! A(X_2, \dots, X_{d+1})$,
- (ii) $D_{X_1} \text{covol}(X_2) = (d+1) \text{covol}(X_2, X_1, \dots, X_1)$,
- (iii) $D_{X_1}^2 \text{covol}(X_2, X_3) = (d+1) d \text{covol}(X_2, X_3, X_1, \dots, X_1)$,
- (iv) $D_{X_1}^d \text{covol}(X_2, \dots, X_{d+1}) = (d+1)! \text{covol}(X_1, \dots, X_{d+1})$,

$$(v) \operatorname{covol}(X_1, \dots, X_{d+1}) = \frac{1}{d+1} \langle X_1, A(X_2, \dots, X_{d+1}) \rangle.$$

Proof. The proof is analogous to the one of Lemma 3.9. \square

COROLLARY 4.11. *For $h_i \in [P]$, $\operatorname{covol}(h_1, \dots, h_{d+1})$ is non-negative.*

Proof. As h_i are support vectors of strongly isomorphic simple polyhedra, the entries of $A(h_2, \dots, h_{d+1})$ are mixed-volume of simple strongly isomorphic Euclidean convex polytopes, hence are non-negative (see Theorem 5.1.6 in [Sch93]). The result follows from (v) because the entries of h_1 are positive. \square

LEMMA 4.12. *For any $h_1, \dots, h_{d-1} \in [P]$, the symmetric bilinear form*

$$\operatorname{covol}(\cdot, \cdot, h_1, \dots, h_{d-1})$$

has trivial kernel.

Proof. The analog of the proof of Lemma 3.13, using Theorem 4.8 instead of Theorem 3.8, gives that in each support hyperplane, the “support vectors” of Z (formally given by (22)) are the ones of a point of \mathbb{R}^d . Let us denote by Z_i the support vector of Z in the hyperplane with normal η_i .

If ϵ is sufficiently small then $h_1 + \epsilon Z$ is the support vector of a Γ -convex polyhedron P_1^ϵ strongly isomorphic to P_1 , the one with support vector h_1 . Moreover the support numbers of the i th facet F_i of P_1^ϵ are the sum of the support numbers of the facet F_i^1 of P_1 with the coefficients of ϵZ_i . As Z_i is the support vector of a point in \mathbb{R}^d , F_i is obtained from F_i^1 by a translation. It follows that each facet of P_1^ϵ is obtained by a translation of the corresponding facet of P_1 , hence P_1^ϵ is a translate of h_1 (the translations of each facet have to coincide on each codimension 2 face). As $h_1 + \epsilon Z$ is supposed to be a Γ -convex polyhedron for ϵ sufficiently small, and as the translation of a Γ -convex polyhedron is not a Γ -convex polyhedron, it follows that $Z = 0$. \square

Theorem 4.13. *For any $h_1, \dots, h_{d-1} \in [P]$, the symmetric bilinear form*

$$\operatorname{covol}(\cdot, \cdot, h_1, \dots, h_{d-1})$$

is positive definite.

Proof. The proof is analogous to the one of Theorem 3.11. \square

Remark on spherical polyhedra. The sets of strongly isomorphic simple Γ -convex polyhedra form convex cones in vector spaces (Lemma 4.9). The mixed-covolume allow to endow these vector spaces with an inner product. Hence, if we restrict to polyhedra of covolume 1, those sets are isometric to convex spherical polyhedra. For $d = 1$ we get simplices named orthoschemes [Fil11b]. In $d = 2$, if we look at the metric induced on the boundary of the Fuchsian polyhedra, we get spherical metrics on subsets of the spaces of flat metrics with cone-singularities of negative curvature on the compact surfaces of genus > 1 . It could be interesting to investigate the shape of these subsets.

5 General Case

5.1 Convexity of the covolume

Hausdorff metric. Recall that $\mathcal{K}(\Gamma)$ is the set of Γ -convex bodies for a given Γ . For K, K' we define the *Hausdorff metric* by

$$d(K, K') = \min\{\lambda \geq 0 \mid K' + \lambda B \subset K, K + \lambda B \subset K'\}.$$

It is not hard to check that this is a distance and that Minkowski sum and multiplication by a positive scalar are continuous for this distance. If we identify Γ -convex bodies with their support functions, then $\mathcal{K}(\Gamma)$ is isometric to a convex cone in $C^0(\mathbb{H}^d/\Gamma)$ endowed with the maximum norm, i.e.:

$$d(K, K') = \sup_{\eta \in \mathbb{H}^d/\Gamma} |h(\eta) - h'(\eta)|.$$

The proof is easy and formally the same as in the Euclidean case [Sch93, 1.8.11].

LEMMA 5.1. *The covolume is a continuous function.*

Proof. Let K be in $\mathcal{K}(\Gamma)$ with support function h . For a given $\epsilon > 0$, choose $\lambda > 1$ such that $(\lambda^{d+1} - 1)\lambda^{d+1}\text{covol}(K) < \epsilon$. Let $\rho < 0$ such that $h > \rho$, and let $\bar{\alpha} > 0$ be the minimum of $h - \rho$. Let $\alpha = \min(\bar{\alpha}, (1 - \lambda)\rho) > 0$. In particular,

$$\rho \leq h - \alpha. \tag{34}$$

Finally, let \bar{K} with support function \bar{h} be such that $d(K, \bar{K}) < \alpha$. In particular, $h - \alpha < \bar{h}$, that, inserted in (34), gives that $\rho < \bar{h}$. This and the definition of α give

$$\bar{h} \leq h + \alpha \leq h + (1 - \lambda)\rho \leq h + (1 - \lambda)\bar{h},$$

i.e. $\lambda\bar{h} \leq h$, i.e. $\lambda\bar{K} \subset K$, in particular $\text{covol}(K) \leq \lambda^{d+1}\text{covol}(\bar{K})$. In a similar way we get $\text{covol}(\bar{K}) \leq \lambda^{d+1}\text{covol}(K)$. This allows to write

$$\begin{aligned} \text{covol}(K) - \text{covol}(\bar{K}) &\leq (\lambda^{d+1} - 1)\text{covol}(\bar{K}) \leq (\lambda^{d+1} - 1)\lambda^{d+1}\text{covol}(K) < \epsilon \\ \text{covol}(\bar{K}) - \text{covol}(K) &\leq (\lambda^{d+1} - 1)\text{covol}(K) \leq (\lambda^{d+1} - 1)\lambda^{d+1}\text{covol}(K) < \epsilon \end{aligned}$$

i.e. $|\text{covol}(K) - \text{covol}(\bar{K})| < \epsilon$. □

The general results are based on polyhedral approximation.

LEMMA 5.2. *Let $K_1, \dots, K_p \in \mathcal{K}(\Gamma)$. There exists a sequence $(P_k^1, \dots, P_k^p)_k$ of strongly isomorphic simple Γ -convex polyhedra converging to (K_1, \dots, K_p) .*

Proof. First, any Γ -convex body K is arbitrarily close to a Γ -convex polyhedron Q . Let Q be the polyhedron made of the orbits of a finite number of support hyperplanes of K . We get $K \subset Q$. For any $\epsilon > 0$, if $Q + \epsilon B$ is not included in K then add facets to Q . The process ends by cocompactness.

Let Q^i be a Γ -convex polyhedron arbitrary close to K_i , and let P be the Γ -convex polyhedron $Q^1 + \dots + Q^p$. Let us suppose that around a vertex x of P , two facets belong to the same orbit for the action of Γ . We perform a little translation in direction of P of a support hyperplane at x , which is not a support hyperplane of a face containing x . A new facet appears, the vertex x disappears, and the two facets in the same orbit share one less vertex. Repeating this operation a finite number of times, we get a polyhedron P' with $N(P') > N(P)$ and such that around each vertex, no facets belong to the same orbit. If P' is not simple, there exists a vertex x of P' such that more than $d + 1$ facets meet at this vertex. We perform a small little parallel move of one of this facets. In this case the number of facets meeting at the vertex x' corresponding to x decreases, and new vertices can appear, but the number of facets meeting at each of those vertices is strictly less than the number of facets meeting at x . If the move is sufficiently small, the number of facets meeting at the other vertices is not greater than it was on P' . Repeating this operation a finite number of times leads to the simple polyhedra P'' , and $N(P'') > N(P')$.

Now we define $P^i = Q^i + \alpha P''$, with $\alpha > 0$ sufficiently small such that P^i remains close to Q^i and hence close to K_i . By definition of P , $N(P) > N(Q^i)$ and finally $N(P'') > N(Q^i)$ hence $N(P^i) = N(P'')$: all the P^i are strongly isomorphic to P'' , which is simple. □

Theorem 5.3. *The covolume is a convex function on the space of Γ -convex bodies: for any $K_1, K_2 \in \mathcal{K}(\Gamma)$, $\forall t \in [0, 1]$,*

$$\text{covol}((1 - t)K_1 + tK_2) \leq t\text{covol}(K_1) + (1 - t)\text{covol}(K_2).$$

Proof. By Lemma 5.2, there exist strongly isomorphic simple Γ -convex polyhedra P_1 and P_2 arbitrary close to respectively K_1 and K_2 . As for simple strongly isomorphic Γ -convex polyhedra, the addition of support vectors is the same as Minkowski addition, Theorem 4.5 gives that

$$\text{covol}((1 - t)P_1 + tP_2) \leq t\text{covol}(P_1) + (1 - t)\text{covol}(P_2)$$

and the theorem follows by continuity of the covolume. □

5.2 Mixed covolume and standard inequalities

LEMMA 5.4. *The covolume on $\mathcal{K}(\Gamma)$ is a homogeneous polynomial of degree $(d + 1)$. Its polar form is the mixed-covolume $\text{covol}(\cdot, \dots, \cdot)$, a continuous non-negative symmetric map on $(\mathcal{K}(\Gamma))^{d+1}$ such that*

$$\text{covol}(K, \dots, K) = \text{covol}(K).$$

Moreover if we restrict to a space of strongly isomorphic simple Γ -convex polyhedra, or to the space of C_+^∞ Γ -convex bodies, then $\text{covol}(\cdot, \dots, \cdot)$ is the same map as the one previously considered.

Proof. Let us define

$$\text{covol}(K_1, \dots, K_{d+1}) = \frac{1}{(d+1)!} \sum_{i=1}^{d+1} (-1)^{d+1+k} \sum_{i_1 < \dots < i_{d+1}} \text{covol}(K_{i_1} + \dots + K_{i_{d+1}}) \quad (35)$$

which is a symmetric map. From the continuity of the covolume and of the Minkowski addition, it is a continuous map. In the case when K_i are strongly isomorphic simple polyhedra, the right-hand side of (35) to the mixed-covolumes previously introduced [Sch93, 5.1.3] (we also could have used another polarization formula [Hör07, (A.5)]). Let us consider a sequence of strongly isomorphic simple Γ -convex polyhedra $P_1(k), \dots, P_{d+1}(k)$ converging to K_1, \dots, K_{d+1} (Lemma 5.2). From the definition of the mixed-covolume we have

$$\text{covol}(\lambda_1 P_1(k) + \dots + \lambda_{d+1} P_{d+1}(k)) = \sum_{i_1, \dots, i_{d+1}=1}^{d+1} \lambda_{i_1} \dots \lambda_{i_{d+1}} \text{covol}(P(k)_{i_1}, \dots, P(k)_{i_{d+1}})$$

and by continuity, passing to the limit,

$$\text{covol}(\lambda_1 K_1 + \dots + \lambda_{d+1} K_{d+1}) = \sum_{i_1, \dots, i_{d+1}=1}^{d+1} \lambda_{i_1} \dots \lambda_{i_{d+1}} \text{covol}(K_{i_1}, \dots, K_{i_{d+1}})$$

so the covolume is a polynomial, and $\text{covol}(\cdot, \dots, \cdot)$ introduced at the beginning of the proof is its polarization. It is non-negative due to Corollary 4.11.

In the case of C_+^2 Γ -convex bodies, both notions of mixed-covolume satisfy (35). \square

Theorem 5.5. *Let $K_i \in \mathcal{K}(\Gamma)$ and $0 < t < 1$. We have the following inequalities.*

Reversed Alexandrov–Fenchel inequality:

$$\text{covol}(K_1, K_2, K_3, \dots, K_{d+1})^2 \leq \text{covol}(K_1, K_1, K_3, \dots, K_{d+1}) \times \text{covol}(K_2, K_2, K_3, \dots, K_{d+1})$$

First reversed Minkowski inequality:

$$\text{covol}(K_1, K_2, \dots, K_2)^{d+1} \leq \text{covol}(K_2)^d \text{covol}(K_1)$$

Second or quadratic reversed Minkowski inequality:

$$\text{covol}(K_1, K_2, \dots, K_2)^2 \leq \text{covol}(K_2) \text{covol}(K_1, K_1, K_2, \dots, K_2)$$

Reversed Brunn–Minkowski inequality:

$$\text{covol}((1-t)K_1 + tK_2)^{\frac{1}{d+1}} \leq (1-t) \text{covol}(K_1)^{\frac{1}{d+1}} + t \text{covol}(K_2)^{\frac{1}{d+1}}$$

Reversed linearized first Minkowski inequality:

$$(d+1) \text{covol}(K_1, K_2, \dots, K_2) \leq d \text{covol}(K_2) + \text{covol}(K_1)$$

If all the K_i are C_+^∞ or strongly isomorphic simple polyhedra, then equality holds in reversed Alexandrov–Fenchel and second reversed Minkowski inequalities if and only if K_1 and K_2 are homothetic.

In the classical case of Euclidean convex bodies, the linearized first Minkowski inequality is valid only on particular subsets of the space of convex bodies, see [Sch93, (6.7.11)].

Proof. Let $P_1(k), \dots, P_{d+1}(k)$ be a sequence of simple strongly isomorphic Γ -convex polyhedra converging to K_1, \dots, K_{d+1} (Lemma 5.2). Applying Cauchy–Schwarz inequality to the inner product $\text{covol}(\cdot, \cdot, P_3(k), \dots, P_{d+1}(k))$ (Theorem 4.13) at $(P_1(k), P_2(k))$ and passing to the limit gives reversed Alexandrov–Fenchel inequality. Equalities cases follow from Theorem 3.11 and 4.13. The second reversed Minkowski inequality and its equality case follows from Alexandrov–Fenchel inequality.

As the covolume is convex (Theorem 5.3), for \bar{K}_1 and \bar{K}_2 of unit covolume, for $\bar{t} \in [0, 1]$ we get

$$\text{covol}((1 - \bar{t})\bar{K}_1 + \bar{t}\bar{K}_2) \leq 1.$$

Taking $\bar{K}_i = K_i / \text{covol}(K_i)^{\frac{1}{d+1}}$ and

$$\bar{t} = \frac{t \text{covol}(K_2)^{\frac{1}{d+1}}}{(1 - t) \text{covol}(K_1)^{\frac{1}{d+1}} + t \text{covol}(K_2)^{\frac{1}{d+1}}}$$

leads to the reversed Brunn–Minkowski inequality.

As $\text{covol}(\cdot)$ is convex, the map

$$f(\lambda) = \text{covol}((1 - \lambda)K_1 + \lambda K_2) - (1 - \lambda)\text{covol}(K_1) - \lambda \text{covol}(K_2), 0 \leq \lambda \leq 1,$$

is convex. As $f(0) = f(1) = 0$, we have $f'(0) \leq 0$, that is the reversed linearized first Minkowski inequality. (Remember that

$$\begin{aligned} \text{covol}((1 - \lambda)K_1 + \lambda K_2) &= (1 - \lambda)^{d+1} \text{covol}(K_1) + (d + 1)(1 - \lambda)^d \lambda \\ &\times \text{covol}(K_1, \dots, K_1, K_2) + \lambda^2 [\dots]. \end{aligned}$$

Reversed Brunn–Minkowski says that the map $\text{covol}(\cdot)^{\frac{1}{d+1}}$ is convex. Doing the same as above with the convex map

$$g(\lambda) = \text{covol}((1 - \lambda)K_1 + \lambda K_2)^{\frac{1}{d+1}} - (1 - \lambda)\text{covol}(K_1)^{\frac{1}{d+1}} - \lambda \text{covol}(K_2)^{\frac{1}{d+1}}, 0 \leq \lambda \leq 1,$$

leads to the first reversed Minkowski inequality. □

The (*Minkowski*) area $S(K)$ of a Γ -convex body K is $(d + 1)\text{covol}(B, K, \dots, K)$. Note that it can be defined from the covolume:

$$S(K) = \lim_{\epsilon \rightarrow 0^+} \frac{\text{covol}(K + \epsilon B) - \text{covol}(K)}{\epsilon}.$$

The following inequality says that, among Γ -convex bodies of area 1, B has smaller covolume, or equivalently that among Γ -convex bodies of covolume 1, B has larger area.

COROLLARY 5.6. (Isoperimetric inequality) *Let K be a Γ -convex body. Then*

$$\left(\frac{S(K)}{S(B)}\right)^{d+1} \leq \left(\frac{\text{covol}(K)}{\text{covol}(B)}\right)^d.$$

Proof. It follows from the first reversed Minkowski with $K_1 = B$, $K_2 = K$, divided by $S(B)^{d+1}$, with $(d+1)\text{covol}(B) = S(B)$. \square

LEMMA 5.7. *If K is a C_+^∞ Γ -convex body, then $S(K)$ is the volume of the Riemannian manifold $\partial K/\Gamma$.*

If K is a Γ -convex polyhedron, then $S(K)$ is the total face area of K (the sum of the area of the facets of K in a fundamental domain).

In particular $S(B)$ is the volume of the compact hyperbolic manifold \mathbb{H}^d/Γ .

Proof. The C_+^2 case follows from the formulas in Section 3, because B is a C_+^2 convex body.

Let K be polyhedral. Let $(P_k)_k$ be a sequence of polyhedra converging to B and such that all the support numbers of P_k are equal to 1 (i.e. all facets are tangent to \mathbb{H}^d). Up to add facets, we can construct P_k such that $N(P_k) > N(K)$ and P_k is simple. Let α be a small positive number. The polyhedron $K + \alpha P_k$ is strongly isomorphic to P_k . It follows from formulas of Section 4 than $(d+1)\text{covol}(P_k, K + \alpha P_k, \dots, K + \alpha P_k)$ is equal to the total face area of $K + \alpha P_k$. By continuity of the mixed-covolume, $(d+1)\text{covol}(P_k, K + \alpha P_k, \dots, K + \alpha P_k)$ converges to $(d+1)\text{covol}(P_k, K, \dots, K)$ when α goes to 0.

We associate to K a support vector $h(K)$ whose entries are support numbers of facets of K , but also to support hyperplanes of K parallel to facets of P_k . We also consider “false faces” of larger codimension, such that the resulting normal fan is the same as the one of P_k . This is possible as the normal fan of P_k is finer than the one of K . The support numbers of the false faces can be computed using (21) and (22) (i.e. K is seen as an element of the closure of $[P_k]$). In particular $h(K + \alpha P_k) = h(K) + \alpha h(P_k)$ and as the map s giving the support numbers of a facet in terms of the support numbers of the polyhedron is linear, the area of this facet is $V_E(s(h(K)) + \alpha s(h(P_k)))$. By continuity of the Euclidean volume, when α goes to 0 this area goes to the area of the facet of K (it is 0 if the facet was a “false facet” of K). Hence $(d+1)\text{covol}(P_k, K, \dots, K)$ is equal to the total face area of K , and on the other hand it goes to $S(K)$ when k goes to infinity. \square

Let us end with an example. Let K be a polyhedral Γ -convex body with support numbers equal to 1. In this case $S(K) = (d+1)\text{covol}(K)$, and as $S(B) = (d+1)\text{covol}(B)$, the isoperimetric inequality becomes

$$\frac{S(K)}{S(B)} \leq 1.$$

Let $d = 2$ and Γ be the Fuchsian group which has a regular octagon as a fundamental domain in the Klein model of \mathbb{H}^2 . Then by the Gauss–Bonnet theorem $S(B) = 4\pi$.

The total face area of K is the area of only one facet, which is eight times the area of a Euclidean triangle of height $h' = \frac{\cosh \varphi - 1}{\sinh \varphi}$ and with edge length two times $h' \frac{1 - \cos \pi/4}{\sin \pi/4}$ (see (21) and (22)). φ is the distance between a point of \mathbb{H}^2 and its image by a generator of Γ , and $\cosh \varphi = 2 + 2\sqrt{2}$ (compare Example C p. 95 in [Kat92] with Lemma 12.1.2 in [MR03]). By a direct computation the isoperimetric inequality becomes

$$0.27 \approx 13 - 9\sqrt{2} \leq \frac{\pi}{2} \approx 1.57.$$

Remarks on equality cases and general Minkowski theorem. Brunn–Minkowski inequality for non-degenerated (convex) bodies in the Euclidean space comes with a description of the equality case. Namely, the equality occurs for a t if and only if the bodies are homothetic (the part “if” is trivial). At a first sight it is not possible to adapt the standard proof of the equality case to the Fuchsian case, as it heavily lies on translations [BF87, Sch93, Ale05].

If such a result was known, it should imply, in a way formally equivalent to the classical one, the characterization of the equality case in the reversed first Minkowski inequality, as well as the uniqueness part in the Minkowski theorem and the equality case in the isoperimetric equality (see below).

The Minkowski problem in the classical case is to find a convex body having a prescribed measure as “area measure” (see notes of Section 5.1 in [Sch93]). It can be solved by approximation (by C_+^2 or polyhedral convex bodies), see [Sch93], or by a variational argument using the volume, see [Ale96]. Both methods require a compactness result, which is known as the Blaschke selection Theorem. Another classical question about Minkowski problem in the C_+^2 case, is to know the regularity of the hypersurface with respect to the regularity of the curvature function, see the survey [TW08]. All those questions can be transposed in the setting of Fuchsian convex bodies.

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